Regularity of weak solutions of the compressible isentropic Navier-Stokes equation

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Abstract

Regularity and uniqueness of weak solution of the compressible isentropic Navier-Stokes equations is proven for small time in dimension $N = 2, 3$ under periodic boundary conditions. In this paper, the initial density is not required to have a positive lower bound and the pressure law is assumed to satisfy a condition that reduces to $\gamma > 1$ when $N = 2, 3$ and $P(\rho) = a\rho^{\gamma}$. In a second part we prove a condition of blow-up in slightly subcritical initial data when $\rho \in L^{\infty}$. We finish by proving that weak solutions in $\mathbb{T}^N$ turn out to be smooth as long as the density remains bounded in $L^{\infty}(L^{(N+1+\epsilon)\gamma})$ with $\epsilon > 0$ arbitrary small.

1 Introduction

The Navier-Stokes equations are the basic model describing the evolution of a viscous compressible gas, As emphasized in many papers related to compressible fluid dynamics [41, 48, 60, 62, 63], vacuum is a major difficulty when trying to prove global existence and strong regularity results. As a matter of fact, starting from initial densities that have positive lower bounds, local existence of smooth solutions can be proved by classical means, since lower bounds on the density persists for small enough time. This paper is devoted to the proof of well-posedness results. We want to prove next that the norm $L^p(L^q)$ on the pressure $P(\rho)$ control the breakdown of strong solutions of the Navier-Stokes equations when $N = 2, 3$. In other words, if a solution of of the Navier-Stokes equations is initially suitably smooth and loses it regularity at some later time, then the maximum norm of the density grows without bounds at the critical time approaches. Let us first recall the periodic compressible isentropic Navier-Stokes equations in $\mathbb{T}^N$ $(N \geq 2)$.

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0 \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^N), \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)Du) - \nabla(\lambda(\rho)\mathrm{div}u) + \nabla P(\rho) = \rho g, \end{cases} \tag{1.1}$$

The unknowns ρ, u respectively correspond to the density of the gas $\rho \geq 0$ and its velocity $u \in \mathbb{R}^N$. The last equation of (1.1) defines the pressure P which is assumed to be increasing in ρ for physical reasons. Usually $P(\rho) = P_{\gamma,a} = a\rho^{\gamma}$ for some positive constant a and some $\gamma \geq 1$. The viscosity coefficients ares assumed to satisfy $\mu > 0$,

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$N\lambda + 2\mu \geq 0$, and the external forces g to belong to $L^2((0,T)\times\mathbb{T}^N)^N$ for all $T>0$. Finally we complement the above system with initial conditions

$$\begin{cases} \rho_{\ t=0} = \rho_0 \geq 0, \\ \rho u_{\ t=0} = m_0. \end{cases} \tag{1.2}$$

Before the remarkable work of P-L Lions, very little was known about solutions of the compressible isentropic Navier-Stokes equation at least when $N\geq 2$. In [50] he proved a global existence theorem and weak stability results for $P_{\gamma,a}$ pressure laws under the following assumptions on the initial data

$$\begin{cases} \rho_0 \in L^1(\mathbb{T}^N)\cap L^\gamma(\mathbb{T}^N), \rho_0 \geq 0, \\ \dfrac{m_0^2}{\rho_0} \in L^1(\mathbb{T}^N), \end{cases} \tag{1.3}$$

where we agree that $\frac{m_0^2}{\rho_0} = 0$ on $\{x\in\mathbb{T}^N$ such that $\rho_0(x)=0\}$. More precisely he proved

Theorem 1.1 *We assume (1.3) and $\gamma>1$ if $N=2$, $\gamma>\frac{3}{2}$ if $N=3$. Then there exists a solution $(\rho,u)\in L^\infty(0,\infty;L^\gamma(\mathbb{T}^N))\times L^2(0,\infty;H^1(\mathbb{T}^N))^N$ satisfying in addition $\rho\in C([0,\infty),L^p(\mathbb{T}^N))$ if $1\leq p<\gamma$, $\rho|u|^2\in L^\infty(0,\infty;L^1(\mathbb{T}^N))$, $\rho\in L^q_{loc}([0,\infty);L^q(\mathbb{T}^N))$ for $1\leq q\leq \gamma-1+\frac{2}{N}\gamma$. Moreover, when $f=0$, for almost all $t\geq 0$, we have*

$$\begin{aligned} \int_{\mathbb{T}^N}(\frac{1}{2}\rho|u|^2+\frac{a}{\gamma-1}\rho^\gamma)(t,x)dx+\int_0^t ds\int_{\mathbb{T}^N}(\mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}u)^2)dxds \\ \leq \int_{\mathbb{T}^N}(\frac{1}{2}\rho_0|u_0|^2+\frac{a}{\gamma-1}\rho_0^\gamma)(x)dx. \end{aligned} \tag{1.4}$$

In addition, he proved similar results for more general pressure laws $P(\rho)$ such that

$$\begin{cases} \displaystyle\int_0^1 \frac{P(s)}{s^2}ds < +\infty, \\ \liminf \to_{s\to+\infty} \dfrac{P(s)}{s^\gamma} > 0, \end{cases} \tag{1.5}$$

for some γ satisfying the above condition of theorem 1.1. Notice that the main difficulty for proving Lions' theorem consists in exhibiting strong compactness properties of the density ρ in $L^p_{loc}(\mathbb{R}^+\times\mathbb{R}^N)$ spaces required to pass to the limit in the pressure term $P(\rho)=a\rho^\gamma$. Let us mention that Feireisl and his collaborators in [28, 29, 30] generalized the result to any $\gamma>\frac{N}{2}$ for $N\geq 2$ in establishing that we can obtain renormalized solution without imposing that $\rho\in L^2_{loc}(\mathbb{R}^+\times\mathbb{R}^N)$ (a property that was needed in Lions' approach in dimension $N=2,3$ giving the further condition $\gamma-1+\frac{2\gamma}{N}\geq 2$), for this he introduces the concept of oscillation defect measure evaluating the loss of compactness. In [7] Bresch and Desjardins show a result of global existence of weak solution for the non isothermal Navier-Stokes system assuming density dependence of $\mu(\rho)$ and $\lambda(\rho)$, considering perfect gas law with some cold pressure close to the vacuum, and the following relation:

$$\lambda(\rho) = 2(\rho\mu'(\rho)-\mu(\rho)). \tag{1.6}$$

The key point in this paper is to show that the structure of the diffusion term provides some regularity for the density thanks to a new mathematical entropy inequality. This one has been discovered in [8], we call it the BD entropy. Mellet and Vasseur by using the BD entropy, get in [52] a very interesting stability result. The interest of this result is to consider conditions where the viscosity coefficients vanish on the vacuum set. It includes the case $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ (when $N = 2$ and $\gamma = 2$, where we recover the Saint-Venant model for Shallow water). The key to the proof is a new energy inequality on the velocity and a gain of integrability, which allows to pass to the limit. Unfortunately, the construction of approximate solutions satisfying: energy estimates, BD mathematical entropy and Mellet-Vasseur estimates is far from being proven except in dimension one or with symmetry assumptions, see [54], [49], [32]. Note that approximate solutions construction process has been proposed in [5] satisfying energy estimates and BD mathematical entropy. This means that only global existence of weak solutions with some extra terms or cold pressure exists in dimension greater than 2.
The existence and uniqueness of local classical solutions for (1.1) with smooth initial data such that the density ρ_0 is bounded and bounded away from zero (i.e., $0 < \underline{\rho} \leq \rho_0 \leq M$) has been stated by Nash in [55]. Let us emphasize that no stability condition was required there.
On the other hand, for small smooth perturbations of a stable equilibrium with constant positive density, global well-posedness has been proved in [51]. Many works in the case of the one dimension have been devoted to the qualitative behavior of solutions for large time (see for example [40, 48]). Refined functional analysis has been used for the last decades, ranging from Sobolev, Besov, Lorentz and Triebel spaces to describe the regularity and long time behavior of solutions to the compressible model [61], [63], [42], [45], [47].
The use of critical functional frameworks led to several new well-posedness results for compressible fluids. In addition to have a norm invariant by (1.1), appropriate functional space for solving (1.1) must provide a control on the L^∞ norm of the density (in order to avoid vacuum and loss of ellipticity). For that reason, we restricted our study to the case where the initial data (ρ_0, u_0) and external force f are such that, for some positive constant $\bar{\rho}$:

$$(\rho_0 - \bar{\rho}) \in B^{\frac{N}{p}}_{p,1},\ u_0 \in B^{\frac{N}{p_1}-1}_{p_1,1} \text{ and } f \in L^1_{loc}(\mathbb{R}^+, \in B^{\frac{N}{p_1}-1}_{p_1,1})$$

with $(p, p_1) \in [1, +\infty[$ good choose. The most important result come from R. Danchin in [22] which show the existence of global solution and uniqueness with initial data close from a equilibrium, and he obtains a similar result in finite time. The interest is that he works in *critical* Besov space (*critical* in the sense of the scaling of the equation). More precisely to speak roughly, he get strong solution with initial data in $B^{\frac{N}{2}}_{2,1} \cap B^{\frac{N}{2}-1}_{2,1} \times (B^{\frac{N}{2}-1}_{2,1})^N$. Here compared with the result on Navier-Stokes incompressible, he needs to control the vacuum and the norm L^∞ of the density in the goal to use the parabolicity of the momentum equation and to have some properties of multiplier spaces. That's why Danchin works in Besov spaces with a third index $r = 1$ for the density, and it's the same for the velocity as the equations are linked. In [25], R. Danchin generalize the previous result with large initial data on the density.
In [25], however, we hand to have $p = p_1$, indeed in this article there exists a very strong

coupling between the pressure and the velocity. To be more precise, the pressure term is considered as a term of rest for the elliptic operator in the momentum equation of (1.1). This paper improve the results of R. Danchin in [22, 25], in the sense that the initial density belongs to larger spaces $B^{\frac{N}{p}}_{p,1}$ with $p \in [1, +\infty[$. The main idea of this paper is to introduce a new variable than the velocity in the goal to *kill* the relation of coupling between the velocity and the density. In [11], F. Charve and R. Danchin and in [14] Q. Chen et al generalize the results from [23] by choosing more general initial data. In particular they works with general Besov space constructed on L^p, however they added some conditions on p ($p < 2N$) to get global solutions. For results of strong solutions with general viscosity coefficients we refer to [15, 33, 34].
In [36], we address the question of local well-posedness in the critical functional framework under the assumption that the initial density belongs to critical Besov space with a index of integrability different of this of the velocity. We adapt the spirit of the results of [1] and [33] which treat the case of Navier-Stokes incompressible with dependent density (at the difference than in these works the velocity and the density are naturally decoupled). The main idea of this paper is to introduce a new variable than the velocity in the goal to "*kill*" the coupling between the velocity and the density. We introduce a new variable v_1 to control the velocity where to avoid the coupling between the density and the velocity, we analyze by a new way the pressure term (in particular we will use this variable v_1 in this article). This idea is inspired from the works of D. Hoff, P-L Lions and D. Serre about the the famous effective pressure. More precisely we write the gradient of the pressure as a Laplacian of the variable v_1, and we introduce this term in the linear part of the momentum equation. We have then a control on v_1 which can write roughly as $u - \mathcal{G}P(\rho)$ where $\mathcal{G}$ is a pseudodifferential operator of order -1. By this way, we have canceled the coupling between v_1 and the density, we next verify easily that we have a control Lipschitz of the gradient of u (it is crucial to estimate the density by the transport equation). This result allows us to reach some critical initial data in the sense that we are not very far to choose $(\rho_0 - \bar{\rho}, u_0)$ in $B^0_{\infty,1} \times B^0_{N,1}$.

On the other hand there have been few existence results on the strong solutions for the general case of nonnegative initial densities. The first result was proved by R. Salvi and I. Straskraba. They showed in [59] that if Ω is a bounded domain, $P = P(\cdot) \in C^2[0, \infty)$, $\rho_0 \in H^2$, $u_0 \in H^1_0 \cap H^2$ and the compatibility condition:

$$Lu_0 + \nabla P(\rho_0) = \rho_0^{\frac{1}{2}} g, \;\; \text{for some } g \in L^2, \tag{1.7}$$

is satisfied, then there exists a unique local strong solution (ρ, u) to the initial boundary value problem (1.1). H. J. Choe and H. Kim proved in [17] a similar existence result when Ω is either a bounded domain or the whole space, $P(\rho) = a\rho^\gamma$ ($a > 0$, $\gamma > 1$), $\rho_0 \in L^1 \cap H^1 \cap W^{1,6}$, $u_0 \in D^1_0 \cap D^2$ and the condition (1.7) is satisfied.
B. Desjardins in [26] proved the local existence of a weak solution solution (ρ, u) with a bounded nonnegative density to the periodic boundary value problem (1.1) as long as $\sup_{0 \leq t \leq T^*}(\|\rho(t)\|_{L^\infty(\mathbb{T}^3)} + \|\nabla u(t)\|_{L^2(\mathbb{T}^3)}) < +\infty$.
This paper is devoted to improve the works in [26] and [17] by choosing very low regularity on the velocity. In the sequel we will note $\frac{d}{dt} = \partial_t + u \cdot \nabla$ and $\dot{f} = \frac{d}{dt} f$.

The viscosity coefficients will suppose constant in the sequel and are assumed to satisfy:

$$\mu > 0, \;\; 0 < \lambda < \frac{5}{4}\mu. \tag{1.8}$$

It follow that there is a $l > 6$, which will be fixed throughout, such that:

$$\frac{\mu}{\lambda} > \frac{(l-2)^2}{4(l-1)}. \tag{1.9}$$

In the sequel we will assume that $g \in E^1_T$ with:

$$\|g\|_{E^1_T} = \|g\|_{L^\infty_T(L^2)} + \|g\|_{L^2_T(L^2)} + \int_0^T (f(s)^7 \|\nabla g\|^2_{L^4} ds + \int_0^T \int_{\mathbb{R}^N} f(s)^5 |\partial_s g|^2 ds dx,$$

where $f(s) = \min(1,s)$. We obtain now the following existence of weak solutions in finite:

Theorem 1.2 *Let $N = 2,3$ and $\gamma \geq 6$. Assume that μ and λ verify (1.8) and (1.9). We assume that $\rho_0 \in L^\infty(\mathbb{R}^N)$, $\rho_0^{\frac{1}{6+\epsilon}} u_0 \in L^{6+\epsilon}$ with $\epsilon > 0$ if $N = 3$, $\rho_0^{\frac{1}{2+\epsilon}} u_0 \in L^{3+\epsilon}$ if $N = 2$ and $\rho_0^{\frac{1}{2}} u_0 \in L^2$. Moreover g is in E^1_T and $g \in L^l(L^l) \cap L^\infty(L^l)$.*

- *There exists $T_0 \in (0, +\infty]$ and a weak solution (ρ, u) to the system (1.1) in $[0, T_0]$ such that for all $T < T_0$ and with $f(t) = \min(t, 1)$, $\omega = \text{curl} u$:*

$$\begin{aligned} &\sup_{0<t\leq T_0} \int_{\mathbb{T}^N} [\frac{1}{2}\rho(t,x)|u(t,x)|^2 + |P(\rho(t,x))| + f(t)|\nabla u(t,x)|^2 dx \\ &+ \sup_{0<t\leq T_0} \int_{\mathbb{T}^N} [\frac{1}{2}\rho(t,x) f(t)^N (\rho|\dot{u}(t,x)|^2 + |\nabla\omega(t,x)|^2) dx \\ &+ \int_0^{T_0} \int_{\mathbb{T}^N} [|\nabla u|^2 + f(s)(\rho|\dot{u}|^2 + |\nabla\omega|^2) + f^N(t)|\nabla\dot{u}|^2] dx dt \leq C C_0. \end{aligned} \tag{1.10}$$

$$\sup_{0<t\leq T_0} \int_{\mathbb{T}^N} \rho(t,x)^{\frac{1}{p}} |u(t,x)|^p dx \leq C_0. \tag{1.11}$$

  where C_0 depends of the initial data ρ_0 and u_0 and where:

$$\begin{cases} p = 2 + \epsilon, & \text{if } N = 2, \\ p = 6 + \epsilon, & \text{if } N = 3. \end{cases} \tag{1.12}$$

- *In addition if we assume that $u_0 \in H^{\frac{N}{2}-1+\epsilon}$ with $\epsilon > 0$ and $\frac{1}{\rho_0} \in L^\infty$, we obtain the following estimates:*

$$\begin{aligned} &\sup_{0<t\leq T_0} \int_{\mathbb{T}^N} [\frac{1}{2}\rho(t,x)|u(t,x)|^2 + |P(\rho(t,x))| + t^{2-\frac{N}{2}-\epsilon}|\nabla u(t,x)|^2 dx \\ &+ \sup_{0<t\leq T_0} \int_{\mathbb{T}^N} [\frac{1}{2}\rho(t,x) t^\sigma (\rho|\dot{u}(t,x)|^2 + |\nabla\omega(t,x)|^2) dx \\ &+ \int_0^{T_0} \int_{\mathbb{T}^N} [|\nabla u|^2 + t^{2-\frac{N}{2}-\epsilon}|\dot{u}|^2 + t^\sigma |\nabla\dot{u}|^2] dx dt \leq C(C_0 + C_f)^\theta, \end{aligned} \tag{1.13}$$

where:

$$\begin{cases} \sigma = 2 - \epsilon, \quad \textit{if } N = 2, \\ \sigma = \dfrac{3}{2} - \epsilon, \quad \textit{if } N = 3, \end{cases} \tag{1.14}$$

and:

$$\sup_{0 \le t \le T_0} \|u(t,\cdot)\|_{H^{\frac{N}{2}-1+\epsilon}} \le C C_0^\theta, \ \textit{ and } \|\frac{1}{\rho} \in L^\infty_{T_0}(L^\infty)., \tag{1.15}$$

and:

$$\nabla u \in L^1_T(BMO). \tag{1.16}$$

- *The regularity properties (1.10), (1.13), (1.15) and (1.16) hold as long as:*

$$\sup_{t \in [0,T]} \|\rho\|_{L^\infty_t(L^\infty(\mathbb{R}^N))} < +\infty. \tag{1.17}$$

Remark 1
- *In this theorem when we assume hypothesis on the vacuum as in [36] where $\frac{1}{\rho_0} \in L^\infty$ we obtain a control on the gradient of the velocity ∇u in $L^1(BMO)$. We know that for incompressible Navier-Stokes this hypothesis is enough to get uniqueness. In this sense we can consider our result as a theorem of strong solutions in a weak sense. It means that this result improve the the results of [36] by the fact that we do not need any other assumption on ρ_0 than $\rho_0 \in L^\infty$. It means that we are completely critical for the scaling of the equations on the density. For the inityial velocity u_0 we need to be a little be surcritical as $u_0 \in H^{\frac{N}{2}-1+\epsilon}$. The only thing is in dimension $n = 3$ where we need extra assumption of the type $u_0 \in L^{6+\epsilon}$.*

- *We can observe that it would be possible to avoid the condition $\gamma \ge 6$ but in this case we would have to ask more integrability on the initial velocity. Indeed this condition plays a crucial role to control the norm L^∞ of the density. In the same spirit in dimension $N = 2$, we can choose initial data slighty surcritical for the scaling of the equations if we choose γ very big.*

- *In this result we improve the works of B. Desjardins in [26] by the fact that we can choose more general initial data. The second important point is that estimates (1.10) and (1.11) can hold as long as $\rho \in L^\infty$ in dimension 2 and 3.*

Let $\bar{\rho} > 0$. In the sequel we will note $q = \rho - \bar{\rho}$. In the following corollary we obtain strong solution by adding some regularity on the initial density and we assume that ρ_0 is away from the vacuum. The goal is to get by this extra regularity on the density a control Lipschitz of ∇u. By this way, we can show that the results of [36] are very critical as it seems necessary to add extra regularity to get a control of ∇u in $L^1_T(L^\infty)$.

Corollary 1 *Under the hypothesis of theorem 1.2 (in particular $u_0 \in H^{\frac{N}{2}-1+\epsilon}$ and there exists $c > 0$ such that $\rho_0 \ge c$). Moreover we assume that $q_0 \in B^\epsilon_{\infty,\infty}$ if $P(\rho) = K\rho$ and $q_0 \in B^{1+\epsilon}_{N,\infty}$ if P is a general pressure. The solutions of theorem 1.2 are then unique and verify locally in time (1.10), (1.13), (1.15) and:*

$$\nabla u \in L^1_{T_0}(L^\infty).$$

Moreover if ρ is in $L^\infty(\mathbb{R} \times \mathbb{T}^N)$ then the solutions are global.

In the following theorem we want improve the criterion of blow-up of corollary 1. More precisely we prove that it is just necessary to control the norm $L^\infty(L^{(N+1+\epsilon)\gamma})$ with $\epsilon>0$ of the density when $P(\rho)=a\rho^\gamma$ with $\gamma\geq 1$ to obtain global strong solution. This result is to connect with the works of Serrin for incompressible Navier-Stokes system where in the compressible Navier-Stokes, the pressure plays the role of the velocity.

Theorem 1.3 *Let $\lambda=0$, $\gamma\geq 1$ and g as in theorem 1.2 and $g\in L^\infty(L^\infty)$. Let $P(\rho)=a\rho^\gamma$ with $a>0$ and $\gamma\geq 1$. Assume that $(q_0,u_0)\in(L^\gamma\cap L^\infty\cap B^{1+\epsilon}_{N,\infty})\times(L^2\cap L^\infty\cap H^{\frac{N}{2}-1+\epsilon})$ with $\epsilon>0$. Moreover ρ_0 check $\rho_0\geq c>0$.*
Let (ρ,u) a global weak solution of system (1.1) on $[0,T)$ with the previous initial data which satisfies the following condition:

$$\begin{aligned}\rho\in L^{\gamma+1}(0,+\infty,L^{(N+1+\epsilon)\gamma})\ \ \textit{and}\ \ \rho\in L^\infty(0,+\infty;L^{9+\epsilon}\cap L^{3\gamma+\frac{3}{2}})\ \ \textit{if}\ \ N=3,\\ \rho\in L^{\gamma+1}(0,+\infty,L^{(N+1+\epsilon)\gamma})\ \ \textit{and}\ \ \rho\in L^\infty(0,+\infty;L^{2\gamma+1}\ \ \textit{if}\ \ N=2,\end{aligned}\tag{1.18}$$

with $\epsilon>0$. Then (ρ,u) is unique and verify locally in time (1.10), (1.13), (1.15) and:

$$\nabla u\in L^1_{loc}(L^\infty).$$

Remark 2

- *When we say that (ρ,u) is unique, we means that (ρ,u) is unique in the following class of global weak solution B_T. B_T is the class of global weak solution (ρ',u') which verifies for the initial data and$\rho_0'\in L^\infty$ and $\sqrt{\rho_0'}u_0'\in L^2$ and:*

$$\rho\in L^\infty(0,\infty,L^\infty)\quad \sqrt{\rho}u\in L^\infty(0,\infty,L^2)\quad \textit{and}\quad \nabla u\in L^2(0,\infty,L^2).$$

- *This result has to be seen as a Prodi-Serrin theorem for compressible Navier-Stokes system. The main difference compared with incompressible Navier-Stokes system is that the good variable is the pressure and not the velocity. In some way, it is the integrability of the pressure which gives the regularity of the solutions. This result is the first one up my knowledge which ask only condition of integrability on the density to get global strong solutions.*

- *In this theorem we can see that by compare with the incompressible Navier-Stokes equation, the good variable to control is not the velocity but the pressure. Indeed if we control enough the pressure, we get integrability on the velocity.*

- *In this theorem we need to assume hypothesis on the vacuum as in [34] where $\frac{1}{\rho_0}\in L^\infty$ in the goal to get a control Lipschitz on the velocity.*

- *This initial data are considered slightly surcritical in dimension 3 except that $u_0\in L^\infty$ (it is probably possible to improve this fact).*

- *Here we need to assume that $\lambda=0$ to get a control L^∞ on u as in the article of A. Mellet and A. Vasseur in [53]. We recall that in this paper they need of a control on $P(\rho)\in L^\infty(L^{3+\epsilon})$ with $\epsilon>0$ for $N=3$ by using some De Giorgi technics used by A. Vasseur in [66] to reprove the famous result of Caffarelli-Kohn and Nirenberg in [10]. As in [53], the pressure plays a important role, and in some sense the pressure is the good variable to control to get global strong solution. The pressure plays the role of the velocity for Navier-Stokes incompressible when we have compressible Navier-Stokes system where a structure of type effective pressure exists.*

- *We can justify the previous statement by seeing the results of D. Bresch and B. Desjardins in [6]. They have a new mathematical entropy for a class of such viscosity coefficients which gives some norms on the gradient of ρ (see [6]), in particular we can obtain the relation (1.18). However this type of viscosity coefficient kill the structure of effective pressure and we can apply our proof. It's again a proof that the structure of the viscosity coefficients plays a crucial role for compressible Navier-Stokes system.*
- *As in the case of incompressible Navier-Stokes system (see [10]), there is a big gap between obtaining (1.18) and by this way have a control on ρ in L^∞.*
- *V. A. Waigant has builded in [67] explicit solutions for which the maximal integrability of the density correspond to $L^q(0,1,L^q)$ with $q=\frac{\gamma(3N+2)-N}{2N}$. It means that (1.18) fails in this case except that the force term is less regular than in our case. It means that the regularity of g is crucial to get strong solution.*

Remark 3 *We believe that our method can be adapted to the euclidian space $\mathbb{R}^N$. This is the object of our future work.*

Our paper is structured as follows. In section 2, we give a few notation and briefly introduce the basic Fourier analysis techniques needed to prove our result. In section 3 and 4, we prove a priori estimate on the density and the velocity. In section 5 and section 6, we prove the theorem 1.2 and corollary 1. We finish in the section 7 by the proof of theorem 1.3.

2 Littlewood-Paley theory and Besov spaces

Throughout the paper, C stands for a constant whose exact meaning depends on the context. The notation $A \lesssim B$ means that $A \leq CB$. For all Banach space X, we denote by $C([0,T],X)$ the set of continuous functions on $[0,T]$ with values in X. For $p\in[1,+\infty]$, the notation $L^p(0,T,X)$ or $L^p_T(X)$ stands for the set of measurable functions on $(0,T)$ with values in X such that $t\rightarrow\|f(t)\|_X$ belongs to $L^p(0,T)$. Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. Let $\varphi\in C^\infty(\mathbb{R}^N)$, supported in $\mathcal{C}=\{\xi\in\mathbb{T}^N/\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$. We set $\mathcal{Q}^N=(0,2\pi)^N$ and $\widetilde{\mathbb{Z}}^N=(\mathbb{Z}/1)^N$. We decompose now $u\in\mathcal{S}'(\mathbb{T})$ into Fourier series:

$$u(x)=\sum_{\beta\in\widetilde{\mathbb{Z}}^N}\hat{u}_\beta e^{i\beta\cdot x}\quad\text{with}\;\;\hat{u}_\beta=\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}}e^{-i\beta\cdot y}u(y)dy.$$

Denoting;

$$h_q(x)=\sum_{\beta\in\widetilde{\mathbb{Z}}^N}\varphi(2^{-q}\beta)e^{i\beta\cdot x},$$

one can now define the periodic dyadyc blocks as:

$$\Delta_q u(x)=\sum_{\beta\in\widetilde{\mathbb{Z}}^N}\varphi(2^{-q}\beta)\hat{u}_\beta e^{i\beta\cdot x},\;\;\text{for all}\;\;q\in\mathbb{Z}$$

and the low frequency cutt-off:

$$S_q u(x) = \hat{u}_0 + \sum_{p \le q-1} \Delta_p u(x).$$

It is obvious that:

$$u = \hat{u}_0 + sum_k \Delta_k u\,.$$

This decomposition is called homogeneous Littlewood-Paley decomposition.

2.1 Homogeneous Besov spaces and first properties

Definition 2.1 *For $s \in \mathbb{R},\ p \in [1, +\infty],\ q \in [1, +\infty],\ $ and $u \in \mathcal{S}'(\mathbb{T}^N)$ we set:*

$$\|u\|_{B^s_{p,q}} = (\sum_{l \in \mathbb{Z}} (2^{ls} \|\Delta_l u\|_{L^p})^q)^{\frac{1}{q}}.$$

The Besov space $B^s_{p,q}$ is the set of temperate distribution u such that $\|u\|_{B^s_{p,q}} < +\infty$.

Remark 4 *The above definition is a natural generalization of the nonhomogeneous Sobolev and Hòlder spaces: one can show that $B^s_{\infty,\infty}$ is the nonhomogeneous Hòlder space C^s and that $B^s_{2,2}$ is the nonhomogeneous space H^s.*

Proposition 2.1 *The following properties holds:*

1. *there exists a constant universal C such that:* $C^{-1} \|u\|_{B^s_{p,r}} \le \|\nabla u\|_{B^{s-1}_{p,r}} \le C \|u\|_{B^s_{p,r}}$.
2. *If $p_1 < p_2$ and $r_1 \le r_2$ then $B^s_{p_1,r_1} \hookrightarrow B^{s-N(1/p_1 - 1/p_2)}_{p_2,r_2}$.*
3. *$B^{s'}_{p,r_1} \hookrightarrow B^s_{p,r}$ if $s' > s$ or if $s = s'$ and $r_1 \le r$.*

Before going further into the paraproduct for Besov spaces, let us state an important proposition.

Proposition 2.2 *Let $s \in \mathbb{R}$ and $1 \le p, r \le +\infty$. Let $(u_q)_{q \ge -1}$ be a sequence of functions such that*

$$(\sum_q 2^{qsr} \|u_q\|^r_{L^p})^{\frac{1}{r}} < +\infty.$$

If $supp\hat{u}_1 \subset \mathcal{C}(0, 2^q R_1, 2^q R_2)$ for some $0 < R_1 < R_2$ then $u = \sum_q u_q$ belongs to $B^s_{p,r}$ and there exists a universal constant C such that:

$$\|u\|_{B^s_{p,r}} \le C^{1+|s|} \big(\sum_{q \ge -1} (2^{qs} \|u_q\|_{L^p})^r \big)^{\frac{1}{r}}.$$

Let now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [4]) and rewrite on a generalized form in [1] by H. Abidi and M. Paicu (in this article the results are written in the case of homogeneous spaces but it can easily generalize for the nonhomogeneous Besov spaces).

Proposition 2.3 *We have the following laws of product:*

- *For all $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ we have:*

$$\|uv\|_{B^s_{p,r}} \leq C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}})\,. \tag{2.19}$$

- *Let $(p,p_1,p_2,r,\lambda_1,\lambda_2) \in [1,+\infty]^2$ such that: $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$, $p_1 \leq \lambda_2$, $p_2 \leq \lambda_1$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{\lambda_1}$ and $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda_2}$. We have then the following inequalities:*
  if $s_1 + s_2 + N\inf(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) > 0$, $s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2}$ then:

$$\|uv\|_{B^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}, \tag{2.20}$$

  when $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ (resp $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$) we replace $\|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}}$) by $\|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,r}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}\cap L^\infty}$), if $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$ we take $r = 1$.
  If $s_1 + s_2 = 0$, $s_1 \in (\frac{N}{\lambda_1} - \frac{N}{p_2}, \frac{N}{p_1} - \frac{N}{\lambda_2}]$ and $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$ then:

$$\|uv\|_{B^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,\infty}} \lesssim \|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,\infty}}. \tag{2.21}$$

  If $|s| < \frac{N}{p}$ for $p \geq 2$ and $-\frac{N}{p'} < s < \frac{N}{p}$ else, we have:

$$\|uv\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}. \tag{2.22}$$

Remark 5 *In the sequel p will be either p_1 or p_2 and in this case $\frac{1}{\lambda} = \frac{1}{p_1} - \frac{1}{p_2}$ if $p_1 \leq p_2$, resp $\frac{1}{\lambda} = \frac{1}{p_2} - \frac{1}{p_1}$ if $p_2 \leq p_1$.*

Corollary 2 *Let $r \in [1,+\infty]$, $1 \leq p \leq p_1 \leq +\infty$ and s such that:*

- *$s \in (-\frac{N}{p_1}, \frac{N}{p_1})$ if $\frac{1}{p} + \frac{1}{p_1} \leq 1$,*
- *$s \in (-\frac{N}{p_1} + N(\frac{1}{p} + \frac{1}{p_1} - 1), \frac{N}{p_1})$ if $\frac{1}{p} + \frac{1}{p_1} > 1$,*

then we have if $u \in B^s_{p,r}$ and $v \in B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty$:

$$\|uv\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}.$$

The study of non stationary PDE's requires space of type $L^\rho(0,T,X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equation through Littlewood-Payley decomposition. But, in doing so, we obtain bounds in spaces which are not type $L^\rho(0,T,X)$ (except if $r = p$). We are now going to define the spaces of Chemin-Lerner in which we will work, which are a refinement of the spaces $L^\rho_T(B^s_{p,r})$.

Definition 2.2 *Let $\rho \in [1,+\infty]$, $T \in [1,+\infty]$ and $s_1 \in \mathbb{R}$. We set:*

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} = \big(\sum_{l\in\mathbb{Z}} 2^{lrs_1} \|\Delta_l u(t)\|^r_{L^\rho(L^p)}\big)^{\frac{1}{r}}.$$

We then define the space $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ as the set of temperate distribution u over $(0,T)\times\mathbb{T}^N$ such that $\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} < +\infty$.

We set $\widetilde{C}_T(\widetilde{B}^{s_1}_{p,r}) = \widetilde{L}^\infty_T(\widetilde{B}^{s_1}_{p,r}) \cap \mathcal{C}([0,T], B^{s_1}_{p,r})$. Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} \leq \|u\|_{L^\rho_T(B^{s_1}_{p,r})} \text{ if } r \geq \rho, \quad \|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} \geq \|u\|_{L^\rho_T(B^{s_1}_{p,r})} \text{ if } r \leq \rho.$$

Remark 6 *It is easy to generalize proposition 2.3, to $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ spaces. The indices s_1, p, r behave just as in the stationary case whereas the time exponent ρ behaves according to Hölder inequality.*

In the sequel we will need of composition lemma in $\widetilde{L}^\rho_T(B^s_{p,r})$ spaces.

Lemma 1 *Let $s > 0$, $(p,r) \in [1,+\infty]$ and $u \in \widetilde{L}^\rho_T(B^s_{p,r}) \cap L^\infty_T(L^\infty)$.*

1. *Let $F \in W^{[s]+2,\infty}_{loc}(\mathbb{T}^N)$ such that $F(0) = 0$. Then $F(u) \in \widetilde{L}^\rho_T(B^s_{p,r})$. More precisely there exists a function C depending only on s, p, r, N and F such that:*

$$\|F(u)\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \leq C(\|u\|_{L^\infty_T(L^\infty)})\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})}.$$

2. *Let $F \in W^{[s]+3,\infty}_{loc}(\mathbb{T}^N)$ such that $F(0) = 0$. Then $F(u) - F'(0)u \in \widetilde{L}^\rho_T(B^s_{p,r})$. More precisely there exists a function C depending only on s, p, r, N and F such that:*

$$\|F(u) - F'(0)u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \leq C(\|u\|_{L^\infty_T(L^\infty)}\|u\|^2_{\widetilde{L}^\rho_T(B^s_{p,r})}.$$

Here we recall a result of interpolation which explains the link of the space $B^s_{p,1}$ with the space $B^s_{p,\infty}$, see [21].

Proposition 2.4 *There exists a constant C such that for all $s \in \mathbb{R}$, $\epsilon > 0$ and $1 \leq p < +\infty$,*

$$\|u\|_{\widetilde{L}^\rho_T(B^s_{p,1})} \leq C\frac{1+\epsilon}{\epsilon}\|u\|_{\widetilde{L}^\rho_T(B^s_{p,\infty})}\bigg(1 + \log\frac{\|u\|_{\widetilde{L}^\rho_T(B^{s+\epsilon}_{p,\infty})}}{\|u\|_{\widetilde{L}^\rho_T(B^s_{p,\infty})}}\bigg).$$

Now we give some result on the behavior of the Besov spaces via some pseudodifferential operator (see [21]).

Definition 2.3 *Let $m \in \mathbb{R}$. A smooth function function $f : \mathbb{T}^N \to \mathbb{R}$ is said to be a $\mathcal{S}^m$ multiplier if for all muti-index α, there exists a constant C_α such that:*

$$\forall \xi \in \mathbb{T}^N, \;\; |\partial^\alpha f(\xi)| \leq C_\alpha(1+|\xi|)^{m-|\alpha|}.$$

Proposition 2.5 *Let $m \in \mathbb{R}$ and f be a $\mathcal{S}^m$ multiplier. Then for all $s \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$ the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

We now focus on the mass equation associated to (1.1)

$$\begin{cases} \partial_t q + u \cdot \nabla q + qh(q) = -q \mathrm{div} v_1, \\ q_{/t=0} = q_0. \end{cases} \tag{2.23}$$

where $h \in C^\infty$, $h(0) = 0$ and $h' \in W^{s,\infty}(\mathbb{R}, \mathbb{R})$. Here v_1 belongs in $\widetilde{L}^1(B^{\frac{N}{p}+\epsilon}_{p,1})$ with $\epsilon > 0$ and $p \in [1, +\infty]$.

Proposition 2.6 *Let $1 \leq p \leq p_1 \leq +\infty$, $p_2 \in [1, +\infty]$ and $1 \leq r \leq +\infty$, $1 \leq r_1 \leq +\infty$. Let assume that:*

$$-N \min\left(\frac{1}{p_1}, \frac{1}{p'}\right) < \sigma \leq \frac{N}{p_2}, \tag{2.24}$$

with strict inequality if $r < +\infty$. Assume that $q_0 \in B^\sigma_{p,r}$, $\nabla v_1 \in L^1(0, T; L^\infty)$, $\mathrm{div} v_1 \in \widetilde{L}^1_T(B^{\frac{N}{p_1}}_{p_1,\infty}) \cap L^\infty$ and that $q \in \widetilde{L}^\infty_T(B^\sigma_{p,r})$ satisfies (2.23). There exists a constant C depending only on N such that for all $t \in [0, T]$ and $m \in \mathbb{Z}$, we have:

$$\|q\|_{\widetilde{L}^\infty_t(B^\sigma_{p,r})} \leq e^{CV(t)} \|q_0\|_{B^\sigma_{p,r}} + e^{CV(t)} - 1, \tag{2.25}$$

with:

$$\begin{cases} V(t) = \displaystyle\int_0^t \left(\|\nabla u(\tau)\|_{B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty} + \|\mathrm{div} v_1(\tau)\|_{B^{\frac{N}{p_2}}_{p_2,\infty} \cap L^\infty} + \|q(\tau)\|^{\alpha+1}_{L^\infty} \right) d\tau \quad \text{if } \sigma < 1 + \frac{N}{p_1}, \\ \quad\;\; = \displaystyle\int_0^t \left(\|\nabla u(\tau)\|_{B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty} + \|\mathrm{div} v_1(\tau)\|_{B^{\frac{N}{p_2}}_{p_2,\infty} \cap L^\infty} + \|q(\tau)\|^{\alpha+1}_{L^\infty} \right) d\tau \quad \text{if } \sigma \leq 1 + \frac{N}{p_1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } r = 1. \end{cases}$$

with α the smallest integer such that $\alpha \geq s$.

Proof: Applying Δ_l to (2.23) yields:

$$\partial_t \Delta_l q + u \cdot \nabla \Delta_l q + \Delta_l q = R_l - \Delta_l(q \mathrm{div} v_1) - \Delta_l(qh(q)) \quad \text{with} \quad R_l = [u \cdot \nabla, \Delta_l] q.$$

Multiplyng by $\Delta_l q |\Delta_l q|^{p-2}$ and performing a time integration, we easily get:

$$\|\Delta_l q(t)\|_{L^p} d\tau \lesssim \|\Delta_l q_0\|_{L^p} + \int_0^t \big(\|R_l\|_{L^p} + \|\mathrm{div} u\|_{L^\infty} \|\Delta_l q\|_{L^p} \\ + \|\Delta_l(q \mathrm{div} v_1)\|_{L^p} + \|\Delta_l(qh(q))\|_{L^p} \big) d\tau.$$

By paraproduct, there exists a constant C and a positive sequence $(c_l) \in l^r$ such that:

$$\|\Delta_l(q \mathrm{div} v_1)\|_{L^p} \leq C c_l 2^{-l\sigma} \|q\|_{B^\sigma_{p,r}} \|\mathrm{div} v_1\|_{B^{\frac{N}{p_2}}_{p_2,\infty} \cap L^\infty}.$$

Similarly by lemma 1 we have:

$$\|\Delta_l(qh(q))\|_{L^p} \leq C c_l 2^{-q\sigma} \|q\|_{B^\sigma_{p,r}} \|q\|^{\alpha+1}_{L^\infty}.$$

Next the term $\|R_l\|_{L^p}$ may be bounded according to the inequality (2.53) p 110 of [3]:

$$\|\big(2^{l\sigma}\|R_l\|_{L^p}\big)_l\|_{l^r} \leq C\|\nabla u\|_{B^{\frac{N}{p_1}}_{p_1,r}}\|q\|_{B^{\sigma}_{p,r}}.$$

We end up with multiplying the previous inequality by $2^{l(\frac{N}{p}+\epsilon)}$ and summing up on $\mathbb{Z}$:

$$\|q(t)\|_{B^{\sigma}_{p,r}} \leq \|q_0\|_{B^{\sigma}_{p,r}} + \int_0^t CV'\|q\|_{B^{\sigma}_{p,r}}d\tau + \int_0^t CV'd\tau.$$

Gronwall lemma yields inequality (2.25). □

3 A priori bounds on the density

In this section we make a formal analysis on the partial differential equations of (1.1) and begin by classical energy estimates. Multiplying the equation of conservation of momentum by u, we obtain

$$\begin{aligned}\int_{\mathbb{T}^N}(\frac{1}{2}\rho|u|^2(t,x)+\Pi(\rho)(t,x)dx+\int_0^t\int_{\mathbb{T}^N}(\mu D(u):D(u)(s,x)\\ +(\lambda+\mu)|\text{div}u|^2(s,x))dsdx \leq \int_{\mathbb{T}^N}\big(\frac{|m_0|^2}{2\rho}(x)+\Pi(\rho_0)(x)]\big)dx,\end{aligned} \tag{3.26}$$

where Π is defined by

$$\Pi(s)=s\bigg(\int_0^s\frac{P(z)}{z^2}dz\bigg), \tag{3.27}$$

It follows classicaly that we have the following bounds

$$\begin{cases}\rho\in L^\infty(0,\infty;L^\gamma_2),\\ \sqrt{\rho}u\in L^\infty(0,\infty;L^2),\\ \nabla u\in L^2(0,\infty;L^2)^{N^2}.\end{cases} \tag{3.28}$$

Here L^γ_2 designes the Orlicz space of definition **??**.

3.1 Bound on $\log\rho$

Let us emphasize that one of the main ingredient of the proof of the first part of theorem 1.2 is a partial differential equation derived from (1.1) involving $\log\rho$. It was introduced by P-L Lions in [50] to prove global existence of weak solutions of (1.1) in a particular case and it is one of the key of the proof in the paper of B. Desjardins in [26]. Letting formally $(\Delta)^{-1}$div operate on the equation of conservation of momentum, we obtain

$$(2\mu+\lambda)\text{div}u-P(\rho)+\int_{\mathbb{T}^N}P(\rho)dx=\partial_t\Delta^{-1}\text{div}(\rho\, u)+R_iR_j(\rho\, u_iu_j) \tag{3.29}$$

where Δ^{-1} denotes the inverse Laplacian with zero mean value on $\mathcal{T}^N$ and R_i the usual Riesz transform. Let us observe that the equation of mass yields

$$\partial_t\log\rho+u.\nabla\log\rho+\text{div}u=0 \tag{3.30}$$

Let us define F and G by the following expression:

$$\begin{aligned} F &= (2\mu+\lambda)(\log\rho + \Delta^{-1}\text{div}(\rho\, u)), \\ G &= (2\mu+\lambda)\text{div}u - P(\rho). \end{aligned}$$

Here F is to compare to a density variable and G is the so-called effective pressure. Moreover we shall denote respectively by P and Q the projection on the space of divergence-free and curl-free vector fields. Combining (3.29) and (2.23), we obtain

$$\partial_t F + u\cdot\nabla F + P(\rho) - \int_{\mathbb{T}^N} P(\rho)dx = [u_j, R_iR_j](\rho u_i). \tag{3.31}$$

Next we define the Lagrangian flow X of u by

$$\begin{cases} \partial_t X(t,s,x) = u(t, X(t,s,x)), \\ X_{/t=s} = x, \end{cases} \tag{3.32}$$

and derive the following identity

$$F(t, X(t,0,x)) = F_0(x) - \int_0^t P(\rho(s,\cdot))dxds + \int_0^t ([u_j, R_iR_j](\rho u_i)(s, X(s,0,x))ds. \tag{3.33}$$

Using the fact that $\rho(\cdot)\geq 0$, we obtain

$$F(t,x) \leq F_0(X(0,t,x)) + \int_0^t P(\rho(s,\cdot))dxds + \int_0^t ([u_j, R_iR_j](\rho u_i)(s, X(s,t,x))ds. \tag{3.34}$$

It follows that

$$\begin{aligned} \log(\rho(t,x)) \leq& \log(\|\rho_0\|_{L^\infty}) + C\|(\Delta)^{-1}\text{div}m_0\|_{L^\infty} + C\|(\Delta)^{-1}\text{div}(\rho u)(t,\cdot)\|_{L^\infty} \\ &+ C_0 t + C\int_0^t \|[u_j, R_iR_j](\rho u_i)(s,\cdot)\|_{L^\infty} ds, \end{aligned} \tag{3.35}$$

where C_0 depends of the initial data. In view of the usual Sobolev embedding inequalities, we obtain

$$\begin{aligned} \log(\rho(t,x)) \leq& \log(\|\rho_0\|_{L^\infty}) + C\|(\Delta)^{-1}\text{div}m_0\|_{L^\infty} + C\|(\Delta)^{-1}\text{div}(\rho u)\|_{L^\infty} \\ &+ C_0 t + C\int_0^t \|[u_j, R_iR_j](\rho u_i)(s,\cdot)\|_{B^1_{N+\epsilon,1}} ds, \end{aligned} \tag{3.36}$$

with $\epsilon > 0$. Let us now remark that we have

$$\|\nabla u\|_{L^{N+\epsilon}} \leq C(\|\text{curlu}\|_{L^{N+\epsilon}} + \|G\|_{L^{N+\epsilon}} + \|P(\rho)\|_{L^{N+\epsilon}}). \tag{3.37}$$

In view of R. Coifman, P-.L. Lions and S. Semmes [19], the following map

$$\begin{aligned} W^{1,r_1}(\mathbb{T}^N)^N \times L^{r_2}(\mathbb{T}^N) &\to W^{1,r_3}(\mathbb{T}^N)^N \\ (a,b) &\to [a_j, R_iR_j]b_i \end{aligned} \tag{3.38}$$

is continuous for any $N \geq 2$ as soon as $\frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}$. Hence we have the following estimates for $N = 3$:

$$\int_0^t \|[u_j, R_iR_j](\rho u_i)(s,\cdot)\|_{L^1(B^1_{N,1})} ds \leq C\|\rho^{\frac{1}{6+\epsilon}} u\|_{L^\infty(L^{6+\epsilon})} \int_0^t \|\nabla u(s)\|_{L^{6-\epsilon}}(1+\|\rho(s)\|_{L^\infty}) ds, \tag{3.39}$$

with $\epsilon > 0$. We obtain finally:

$$\begin{aligned} \log(\rho(t,x)) \leq & \log(\|\rho_0\|_{L^\infty}) + C\|(\Delta)^{-1}\text{div}m_0\|_{L^\infty} + C\|(\Delta)^{-1}\text{div}(\rho u)\|_{L^\infty} \\ & + C_0 t + C\|\nabla u\|_{L^2_t(L^6)} \|\rho u\|_{L^2_t(L^{6+\epsilon})}, \end{aligned} \tag{3.40}$$

4 A priori estimates for the velocity

4.1 Gain of integrability of the velocity u

We want here derive estimate of integrability on the velocity u. This idea has been successively used in different papers, we refer in particularly to [43] and [52, 53]. To do it, we multiply the momentum equation by $u|u|^{p_1-2}$ and we get after integration by part:

$$\begin{aligned} \frac{1}{p_1}\int_{\mathbb{T}^N} \rho|u|^{p_1}(t,x)dx + \mu\int_0^t\int_{\mathbb{T}^N} \big(|u|^{p_1-2}|\nabla u|^2(t,x) + \frac{p_1-2}{4}|u|^{p_1-4}|\nabla|u|^2|^2(t,x)\big)dxdt \\ + \lambda\int_0^t\int_{\mathbb{T}^N}\big((\text{div}u)^2|u|^{p_1-2}(t,x) + \frac{p_1-2}{2}\text{div}u\sum_i u_i\partial_i|u|^2|u|^{p_1-4}(t,x)\big)dtdx \\ - \int_0^t\int_{\mathbb{T}^N} P(\rho)\big(\text{div}u|u|^{p_1-2} + (p_1-2)\sum_{i,k} u_iu_k\partial_i u_k|u|^{p_1-4}\big)(t,x)dtdx \\ \leq \int_{\mathbb{T}^N}\rho_0|u_0|^{p_1}dx. \end{aligned}$$

We have then by Young's inequality:

$$\begin{aligned} & \frac{\lambda(p_1-2)}{2}\int_0^t\int_{\mathbb{T}^N}\text{div}u\sum_i u_i\partial_i|u|^2|u|^{p_1-4}(t,x)\big)dtdx \\ & = \lambda\frac{p_1-2}{2}\int_0^t\int_{\mathbb{T}^N}\text{div}u\, u\cdot\nabla(|u|^2)|u|^{p_1-4}(t,x)dtdx \leq \\ & \lambda\frac{p_1-2}{2}(\frac{\eta}{2}\int_0^t\int_{\mathbb{T}^N}|\text{div}u|^2|u|^{p_1-2}dtdx + \frac{2}{\eta}\int_0^t\int_{\mathbb{T}^N}|\nabla|u|^2|^2|u|^{p_1-4}(t,x)dtdx) \end{aligned}$$

If we choose:

$$\lambda\frac{\eta(p_1-2)\lambda}{4} = s\mu + \lambda,$$

for some $s \in (0, \frac{1}{N})$,by the fact that $(\text{div}u)^2 \leq N|\nabla u|^2$ we therefore obtain:

$$\begin{aligned} \frac{1}{p_1}\int_{\mathbb{T}^N}\rho|u|^{p_1}(t,x)dx + A_s\int_0^t\int_{\mathbb{T}^N}|u|^{p_1-2}|\nabla u|^2(t,x)dtdx \\ + B_s\int_0^t\int_{\mathbb{T}^N}|u|^{p_1-4}|\nabla|u|^2|^2(t,x)dxdt \leq \int_0^t\int_{\mathbb{T}^N}P(\rho)\big(\text{div}u|u|^{p_1-2} \\ + \frac{p_1-2}{2}u\cdot\nabla(|u|^2)|u|^{p_1-4}\big)(t,x)dtdx + \int_{\mathbb{T}^N}\rho_0|u_0|^{p_1}dx, \end{aligned}$$

with $A_s=\mu(1-sN)$ and $B(s)=\frac{p_1-2}{4}\mu-\frac{(p_1-2)^2\lambda^2}{16(s\mu+\lambda)}$. By Young inequality, we get again:

$$\begin{aligned}&\frac{1}{p_1}\int_{\mathbb{T}^N}\rho|u|^{p_1}(t,x)dx+A_s\int_0^t\int_{\mathbb{T}^N}|u|^{p_1-2}|\nabla u|^2(t,x)dtdx\\&+B_s\int_0^t\int_{\mathbb{T}^N}|u|^{p_1-4}|\nabla|u|^2|^2(t,x)dxdt\leq C_\epsilon\int_0^t\int_{\mathbb{T}^N}P(\rho)^2|u|^{p_1-2}dtdx+\int_{\mathbb{T}^N}\rho_0|u_0|^{p_1}dx,\end{aligned}$$

with C_ϵ enough big. Now we want use the fact that $\nabla|u|^{\frac{p_1}{2}}\in L^2_t(L^2)$, which implies that when $N=3$ $u\in L^{p_1}_t(L^{3p_1})$. More precisely we have:

$$\|u\|^{\frac{p_1}{2}}_{L^{p_1}_t(L^{3p_1})}\leq C(\|\nabla(|u|^{\frac{p_1}{2}})\|_{L^2(L^2)}+\bar{u}_{\frac{p_1}{2}}),$$

where $\bar{u}_{\frac{p_1}{2}})$ is the average of $|u|^{\frac{p_1}{2}}$. We have then by Hölder's inequalities with $\frac{p_1-2}{3p_1}+\frac{2(p_1+1)}{3p_1}=1$ and $\frac{p_1-2}{p_1}+\frac{2}{p_1}=1$:

$$\begin{aligned}|\int_0^t\int_{\mathbb{T}^N}P(\rho)^2|u|^{p_1-2}dtdx|&\leq\|P(\rho)^2\|_{L_t^{\frac{p_1}{2}}(L^{\frac{3p_1}{2(p_1+1)}})}\||u|^{p_1-2}\|_{L_t^{\frac{p_1}{p_1-2}}(L^{\frac{3p_1}{p_1-2}})},\\&\leq\|P(\rho)\|^2_{L^{p_1}_t(L^{\frac{3p_1}{p_1+1}})}\|u\|^{p_1-2}_{L^{p_1}_t(L^{3p_1})},\\&\leq C\|P(\rho)\|^2_{L^{p_1}_t(L^{\frac{3p_1}{p_1+1}})}(\|\nabla(|u|^{\frac{p_1}{2}})+\widetilde{|u|^{\frac{p_1}{2}}}(\cdot)\|_{L^2_t(L^2)})^{2-\frac{4}{p_1}}.\end{aligned}$$

Remarking that $\int_{\mathbb{T}^N}\rho_0dx=M\neq0$, we can write as $\gamma\geq\frac{6}{5}$:

$$\begin{aligned}&\widetilde{|u|^{\frac{p_1}{2}}}(s)\leq\frac{1}{M}(\|\rho(s,\cdot)\|_{L^\gamma}\|\nabla|u|^{\frac{p_1}{2}}(s,\cdot)\|_{L^2}+\int_{\mathbb{T}^N}\rho(s,x)|u(s,x)|^{\frac{p_1}{2}}dx),\\&(\int_0^t\widetilde{|u|^{\frac{p_1}{2}}}^2(s)ds)^{\frac{1}{2}}\leq\frac{1}{M}(\|\rho\|_{L^\infty_t(L^\gamma)}\|\nabla|u|^{\frac{p_1}{2}}\|_{L^2_t(L^2)}+(Mt)^{\frac{1}{2}}\|\rho^{\frac{1}{p_1}}u\|^{\frac{p_1}{2}}_{L^\infty_t(L^{p_1})}),\end{aligned}$$

We obtain then:

$$\begin{aligned}&\|P(\rho)\|^2_{L^{p_1}_t(L^{\frac{3p_1}{p_1+1}})}(\|\nabla(|u|^{\frac{p_1}{2}})+\widetilde{|u|^{\frac{p_1}{2}}}(\cdot)\|_{L^2_t(L^2)})^{2-\frac{4}{p_1}}\leq\\&C_1\|P(\rho)\|^2_{L^{p_1}_t(L^{\frac{3p_1}{p_1+1}})}\big(\|\nabla(|u|^{\frac{p_1}{2}})\|^{\frac{2p_1-4}{p_1}}_{L^2_t(L^2)}+(Mt)^{\frac{p_1-2}{p_1}}\|\rho^{\frac{1}{p_1}}u\|^{\frac{p_1-2}{2}}_{L^\infty_t(L^{p_1})}\big)\end{aligned}$$

By a standard application of Young inequality ($\frac{2p_1-4}{2p_1}+\frac{4}{2p_1}=1$), we obtain that:

$$\begin{aligned}&\frac{1}{p_1}\int_{\mathbb{T}^N}\rho|u|^{p_1}(t,x)dx+A_s\int_0^t|u|^{p_1-2}|\nabla u|^2(t,x)dtdx\\&+B_s\int_0^t|u|^{p_1-4}|\nabla|u|^2|^2(t,x)dxdt\leq C^2_{\epsilon,t}\|P(\rho)\|^{2p_1}_{L^{p_1}_t(L^{\frac{3p_1}{p_1+1}})}+\frac{1}{p_1}\int_{\mathbb{T}^N}\rho_0|u_0|^{p_1}dx,\end{aligned}\tag{4.41}$$

where $C_{\epsilon,t}$ is big enough and depend of the time t.

Case $N = 2$

By proceeding similarly, we have for all $q > 1$ in particular q big:

$$\|u\|^{\frac{p_1}{2}}_{L^{p_1}(L^{\frac{p_1 q}{2}})} \leq C(\|\nabla |u|^{\frac{p_1}{2}}\|_{L^2(L^2)} + \bar{u}_{\frac{p_1}{2}}),$$

We have then by Hölder's inequalities with $\frac{2(p_1-2)}{qp_1} + \frac{(q-2)p_1+4)}{qp_1} = 1$ and $\frac{p_1-2}{p_1} + \frac{2}{p_1} = 1$:

$$\begin{aligned}
|\int_0^t \int_{\mathbb{T}^N} P(\rho)^2 |u|^{p_1-2} dt dx| &\leq \|P(\rho)^2\|_{L_t^{\frac{p_1}{2}}(L^{\frac{qp_1}{(q-2)p_1+4}})} \| |u|^{p_1-2}\|_{L_t^{\frac{p_1}{p_1-2}}(L^{\frac{qp_1}{2(p_1-2)}})}, \\
&\leq \|P(\rho)\|^2_{L_t^{p_1}(L^{\frac{2qp_1}{(q-2)p_1+4}})} \|u\|^{p_1-2}_{L_t^{p_1}(L^{qp_1})}, \\
&\leq C\|P(\rho)\|^2_{L_t^{p_1}(L^{\frac{2qp_1}{(q-2)p_1+4}})} (\|\nabla(|u|^{\frac{p_1}{2}}) + \widetilde{|u|^{\frac{p_1}{2}}}(\cdot)\|_{L_t^2(L^2)})^{2-\frac{4}{p_1}}.
\end{aligned}$$

and so:

$$\begin{aligned}
&\frac{1}{p_1}\int_{\mathbb{T}^N} \rho |u|^{p_1}(t,x) dx + A_s \int_0^t |u|^{p_1-2} |\nabla u|^2 (t,x) dt dx \\
&+ B_s \int_0^t |u|^{p_1-4} |\nabla |u|^2|^2 (t,x) dx dt \leq C^2_{\epsilon,t} \|P(\rho)\|^2_{L_t^{p_1}(L^{\frac{2qp_1}{(q-2)p_1+4}})} + \frac{1}{p_1}\int_{\mathbb{T}^N} \rho_0 |u_0|^{p_1} dx,
\end{aligned} \tag{4.42}$$

4.2 Gain of derivatives on the velocity u

In this section we deal with the case $N = 3$. The case $N = 2$ follows the same lines. In the sequel we will follow the procedure developped in [26] and [43] to get some energy inequalities. The main idea compared with the results in [16, 17, 18] is to obtain energy inequalities which depends only of the control on $\rho \in L^\infty$. It implies that we have to be carreful to not introduce some derivatives on the density in the goal to "kill" the coupling between velocity and pressure. Multiplying first the equation of conservation of momentum by $f(t)\partial_t u$ with $f(t) = \min(1,t)$ and integrating over $(0,T) \times \mathbb{T}^N$, we deduce that:

$$\begin{aligned}
&\int_0^t \int_{\mathbb{T}^N} f(t)\rho |\partial_t u|^2 dx ds + \frac{1}{2}\int_{\mathbb{T}^N} f(t)\big(\mu |\nabla u(t,x)|^2 + (\lambda+\mu)|\mathrm{div} u|^2(t,x)\big) dx \\
&+ \int_0^t \int_{\mathbb{T}^N} \nabla P(\rho)\cdot f(t)\partial_t u dx ds \leq \int_0^t \int_{\mathbb{T}^N} \frac{f'(t)}{2}\big(\mu|\nabla u(t,x)|^2 + (\lambda+\mu)|\mathrm{div} u|^2(t,x)\big) dx ds \\
&\qquad + \int_0^t \|\sqrt{f(t)\rho}\,\partial_t u\|_{L^2(\mathbb{T}^N)} (\|\sqrt{f(t)\rho}\,(u\cdot\nabla)u\|_{L^2(\mathbb{T}^N)} + \|\sqrt{\rho} g\|_{L^2(\mathbb{T}^N)}) ds.
\end{aligned} \tag{4.43}$$

Next we use the equation of mass conservation to write:

$$\begin{aligned}&\int_{\mathbb{T}^N}\nabla P(\rho)\cdot f(t)\partial_t u dxds=-\int_{\mathbb{T}^N}P(\rho)f(t)\partial_t \text{div} u dxds,\\&=-\partial_t\int_{\mathbb{T}^N}f(t)P(\rho)\text{div}u dx+\int_{\mathbb{T}^N}\partial_t(f(t)P(\rho))\text{div}u dx\\&=-\partial_t\int_{\mathbb{T}^N}f(t)P(\rho)\text{div}u dx-\int_{\mathbb{T}^N}f(t)\big[\text{div}(P(\rho)u)\text{div}u+(\rho P^{'}(\rho)-P(\rho))\text{div}u\big]dx\\&\qquad\qquad+\int_{\mathbb{T}^N}f^{'}(t)P(\rho)\text{div}u dx,\\&=-\partial_t\int_{\mathbb{T}^N}f(t)P(\rho)\text{div}u dx+\frac{1}{2\mu+\lambda}\int_{\mathbb{T}^N}f(t)P(\rho)u\cdot\nabla(G+P(\rho))dx\\&-\frac{1}{(2\mu+\lambda)^2}\int_{\mathbb{T}^N}f(t)(\rho P^{'}(\rho)-P(\rho))\big(G^2-P(\rho)^2+2(\lambda+2\mu)P(\rho)\text{div}u\big)dx\\&\qquad\qquad+\int_{\mathbb{T}^N}f^{'}(t)P(\rho)\text{div}u dx,\end{aligned}$$

If we define $\Pi_f(s)=s(\int_0^s\frac{f(z)}{z^2}dz)$, we have then by mass equation:

$$\partial_t\Pi_f(\rho)+\text{div}(\Pi_f(\rho)u)+f(\rho)\text{div}u=0.$$

By this fact we obtain that:

$$P(\rho)\big(u\cdot\nabla P(\rho)-2(\rho P^{'}(\rho)-P(\rho))\text{div}u\big)=P(\rho)(-\partial_t(P(\rho))-3P^{'}(\rho)\rho\text{div}u+2P(\rho)\text{div}u).$$

We have then:

$$\begin{aligned}&\int_{\mathbb{T}^N}P(\rho)\big(u\cdot\nabla P(\rho)-2(\rho P^{'}(\rho)-P(\rho))\text{div}u\big)dx=\\&-\frac{1}{2}\int_{\mathbb{T}^N}\partial_t(P(\rho)^2)dx+3\int_{\mathbb{T}^N}\partial_t\Pi_{3P^{'}(\rho)\rho}dx-2\int_{\mathbb{T}^N}\partial_t\Pi_{P(\rho)}dx.\end{aligned}$$

Next by integration by parts, we have:

$$\Pi_{3P^{'}(s)s}=\frac{1}{2}P(s)^2+\frac{s}{2}\int_0^s\frac{P(z)^2}{z^2}dz.$$

So:

$$\int_{\mathbb{T}^N}P(\rho)\big(u\cdot\nabla P(\rho)-2(\rho P^{'}(\rho)-P(\rho))\text{div}u\big)dx=\int_{\mathbb{T}^N}\partial_t(P(\rho)^2-\Pi_{P(\rho)})dx.$$

We set $k(s)=P(s)^2-\frac{1}{2}\Pi_{P(s)}$. Let us observe that in the $P_{a,\gamma}$ case, we have $k(s)=a^2s^{2\gamma}((2\gamma-\frac{3}{2})/(2\gamma-1))$.
We obtain finally:

$$\begin{aligned}&\int_{\mathbb{T}^N}\nabla P(\rho)\cdot f(t)\partial_t u dx=-\partial_t\int_{\mathbb{T}^N}f(t)P(\rho)\text{div}u dx+\frac{1}{\lambda+2\mu}\partial_t\int_{\mathbb{T}^N}f(t)k(\rho)dx\\&+\frac{1}{2\mu+\lambda}\int_{\mathbb{T}^N}f(t)P(\rho)u\cdot\nabla G dx+\frac{1}{(2\mu+\lambda)^2}\int_{\mathbb{T}^N}f(t)P^2(\rho)(\rho P^{'}(\rho)-P(\rho))dx\\&\qquad-\frac{1}{(2\mu+\lambda)^2}\int_{\mathbb{T}^N}f(t)G^2(\rho P^{'}(\rho)-P(\rho))dx-\frac{1}{\lambda+2\mu}\int_{\mathbb{T}^N}f^{'}(t)k(\rho)dx\\&\qquad\qquad+\int_{\mathbb{T}^N}f^{'}(t)P(\rho)\text{div}u dx.\end{aligned}$$

Inserting the above inequality in (4.43) and by Young's inequality, we obtain:

$$\begin{aligned}
&\int_0^t\int_{\mathbb{T}^N} f(t)\rho|\partial_t u|^2 dxds+\frac{1}{2}\int_{\mathbb{T}^N} f(t)\big(\mu|\nabla u(t,x)|^2+(\lambda+\mu)(\text{div}u(t,x))^2\big)dx\\
&\quad+\frac{1}{(2\mu+\lambda)^2}\int_0^t\int_{\mathbb{T}^N} f(t)P(\rho)^2(\rho P^{'}(\rho)-P(\rho))dxds\\
&\quad+\frac{1}{\lambda+2\mu}\int_{\mathbb{T}^N} f(t)k(\rho(t,x))dx\leq C+\int_{\mathbb{T}^N} f(t)P(\rho(t,x))\text{div}u(t,x)dx\\
&\quad+\frac{1}{\lambda+2\mu}\int_0^t\int_{\mathbb{T}^N} f^{'}(t)k(\rho)dx-\int_0^t\int_{\mathbb{T}^N} f^{'}(t)P(\rho)\text{div}udx\\
&\quad+C\int_0^t\int_{\mathbb{T}^N} f(t)(|\rho P^{'}(\rho)-P(\rho)|G^2+|P(\rho)u||\nabla G|+|\sqrt{\rho}u\cdot\nabla u|^2\\
&\quad+|\sqrt{\rho}g|^2)dxds.
\end{aligned}\tag{4.44}$$

In the sequel we set:

$$\begin{aligned}
A(t)=&\int_0^t\int_{\mathbb{T}^N} f(t)\rho|\partial_t u|^2 dxds+\frac{1}{2}\int_{\mathbb{T}^N} f(t)\big(\mu|\nabla u(t,x)|^2+(\lambda+\mu)(\text{div}u(t,x))^2\big)dx\\
&+\frac{1}{(2\mu+\lambda)^2}\int_0^t\int_{\mathbb{T}^N} f(s)P(\rho)^2(\rho P^{'}(\rho)-P(\rho))dxds+\frac{1}{\lambda+2\mu}\int_{\mathbb{T}^N} f(t)k(\rho(t,x))dx
\end{aligned}$$

We obtain finally:

$$\begin{aligned}
A(t)&\leq C+C_t\|\rho\|_{L^\infty}+C\int_0^t\int_{\mathbb{T}^N} f(s)(|\rho P^{'}(\rho)-P(\rho)|G^2+|P(\rho)u||\nabla G|\\
&\qquad+|\sqrt{\rho}u\cdot\nabla u|^2+|\sqrt{\rho}g|^2)dxds.\\
&\leq C+C\int_0^t(\|\rho P^{'}(\rho)-P(\rho)\|_{L^\infty}\|\sqrt{f(s)}G\|^2_{L^2}+f(s)\|g(\rho)(s,\cdot)\|_{L^\infty}\|\sqrt{\rho}u\|_{L^2}\\
&\qquad\times\|\nabla G\|_{L^2}+\|\sqrt{\rho}u\|^2_{L^4}\|\sqrt{f(s)}\nabla u\|^2_{L^4}+\|\rho\|_{L^\infty}\|\sqrt{f(s)}g\|^2_{L^2})dxds,\\
&\leq C+C\int_0^t(\|h(\rho(s,\cdot)\|_{L^\infty}\|\sqrt{f(s)}\nabla u\|^2_{L^2}+f(s)\|i(\rho(s,\cdot)\|_{L^\infty}+f(s)\|\nabla G\|_{L^2}\\
&\qquad\times\|k(\rho(s,\cdot))\|_{L^\infty}+\|\sqrt{\rho}u\|^2_{L^4}\|\sqrt{f(s)}\nabla u\|^2_{L^4}+\|\rho\|_{L^\infty}\|\sqrt{f(s)}g\|^2_{L^2})dxds,
\end{aligned}\tag{4.45}$$

where $k(\rho)=\frac{P(\rho)}{\sqrt{\rho}}$, $h(s)=|sP^{'}(s)-P(s)|$ and $i(s)=h(s)P(s)^2$.

Estimates on Pu and G

We want now to obtains bounds on $\mathcal{P}u$ and g, assuming that ρ is a priori bounded in $L^\infty(\mathbb{T}^N)$. Indeed we wnat show that the control of $A(t)$ in (4.45) depend only of a control on $\|\rho\|_{L^\infty}$.
Next we use once more the equation of conservation of momentum to write:

$$\mu\Delta u+(\lambda+\mu)\nabla\text{div}u=\mathcal{P}(\rho\partial_t u)+\mathcal{P}(\rho u\cdot\nabla u)-\mathcal{P}(\rho g),\tag{4.46}$$

$$\nabla G=\mathcal{Q}(\rho\partial_t u)+\mathcal{Q}(\rho u\cdot\nabla u)-\mathcal{Q}(\rho g).\tag{4.47}$$

We recall here that $\mathcal{P}$ is the projector on free-divergence vector field and $\mathcal{Q}$ is the projector on gradient vector field. Therefore we have:

$$\begin{aligned}\|\nabla G\|_{L^2}+\|\Delta\mathcal{P}u\|_{L^2}\leq C\|\rho(s,\cdot)\|_{L^\infty}^{\frac{1}{2}}\big(\|\sqrt{\rho}\partial_s u(s\cdot)\|_{L^2}+\|\sqrt{\rho}u\cdot\nabla u(s,\cdot)\|_{L^2}\\+\|\rho(s,\cdot)\|_{L^\infty}^{\frac{1}{2}}\|g(s,\cdot)\|_{L^2}\big).\end{aligned}\tag{4.48}$$

The case $N=3$

For simplicity, we will treat only the case of the dimension 3. We recall that for all $1<p<+\infty$:

$$\|\nabla u\|_{L^p}\leq C(\|\nabla\mathcal{P}u\|_{L^p}+\|RG\|_{L^p}+\|R(P(\rho))\|_{L^p}).$$

where R is a pseudo differential operator of order 0 such that for all $f\in H^1(\mathbb{T}^N)$ $\int_{\mathbb{T}^N}Rf dx=0$. We want now to recall the Gagliardo-Nirenberg's theorem:

$$\forall f\in H^1(\mathbb{T}^N)\;\text{ such that }\;\int_{\mathbb{T}^N}f dx=0,\;\;\|f\|^2_{L^4(\mathbb{T}^N)}\leq C\|f\|^{\frac{1}{2}}_{L^2(\mathbb{T}^N)}\|\nabla f\|^{\frac{3}{2}}_{L^2(\mathbb{T}^N)}.$$

We deduce that from Gagliardo-Nirenberg's inequality, Young's inequalities and (4.48):

$$\begin{aligned}&\|\sqrt{\rho}u\|^2_{L^4}\|\sqrt{f(s)}\nabla u\|^2_{L^4}\leq Cf(s)\|\sqrt{\rho}u\|^2_{L^4}(\|R(P(\rho))\|^2_{L^4}+\|\nabla\mathcal{P}u\|_{L^4}+\|RG\|_{L^4}\big)\\&\leq C\|\sqrt{\rho}u\|^2_{L^4}\big(f(s)\|P(\rho)\|^2_{L^4}+\big(\sqrt{f(s)}(\|\nabla u\|_{L^2}+\|P(\rho)\|_{L^2})\big)^{\frac{1}{2}}\\&\qquad\qquad\times\big(\sqrt{f(s)}(\|\Delta\mathcal{P}u\|_{L^2}+\|\nabla G\|_{L^2})\big)^{\frac{3}{2}}\big)\\&\leq C\big(f(s)\|\sqrt{\rho}u\|^2_{L^4}\|P(\rho)\|^2_{L^4}+\|\rho(s,\cdot)\|^{\frac{3}{4}}_{L^\infty}\|\sqrt{\rho}u\|^2_{L^4}\big(\sqrt{f(s)}(\|\nabla u\|_{L^2}+\|P(\rho)\|_{L^2})\big)^{\frac{1}{2}}\\&\times\big(\sqrt{f(s)}(\|\sqrt{\rho}\partial_s u(s\cdot)\|_{L^2}+\|\sqrt{\rho}u\cdot\nabla u(s,\cdot)\|_{L^2}+\|\rho(s,\cdot)\|^{\frac{1}{2}}_{L^\infty}\|f(s,\cdot)\|_{L^2}\big)^{\frac{3}{2}}\big)\\&\leq C\big(f(s)\|\sqrt{\rho}u\|^2_{L^4}\|P(\rho)\|^2_{L^4}+\frac{1}{\epsilon}f(s)(\|\nabla u\|^2_{L^2}+\|P(\rho)\|^2_{L^2})\|\rho(s,\cdot)\|^3_{L^\infty}\|\sqrt{\rho}u\|^8_{L^4}\\&+\epsilon f(s)(\|\sqrt{\rho}\partial_s u(s\cdot)\|^2_{L^2}+\|\sqrt{\rho}u\cdot\nabla u(s,\cdot)\|^2_{L^2}+\|\rho(s,\cdot)\|_{L^\infty}\|g(s,\cdot)\|^2_{L^2})\big).\end{aligned}\tag{4.49}$$

Hence we obtain by Young inequality from (4.48):

$$\begin{aligned}&f(s)\|k(\rho(s,\cdot)\|_{L^\infty}\|\nabla G(s,\cdot)\|_{L^2}\leq\frac{C}{\epsilon}\|k(\rho(s,\cdot)\|^2_{L^\infty}\|\rho(s,\cdot)\|_{L^\infty}f(s)\\&+\epsilon(\|\sqrt{f(s)\rho}\partial_s u(s\cdot)\|^2_{L^2}+\|\sqrt{\rho f(s)}u\cdot\nabla u(s,\cdot)\|^2_{L^2}+f(s)\|\rho(s,\cdot)\|_{L^\infty}\|g(s,\cdot)\|^2_{L^2}).\end{aligned}\tag{4.50}$$

By adding (4.49) and (4.50), we obtain:

$$\begin{aligned}&\|\sqrt{\rho}u\|^2_{L^4}\|\sqrt{f(s)}\nabla u\|^2_{L^4}+f(s)\|k(\rho(s,\cdot)\|_{L^\infty}\|\nabla G(s,\cdot)\|_{L^2}\leq\\&C\big(f(s)\|\sqrt{\rho}u\|^2_{L^4}\|P(\rho)\|^2_{L^4}+\frac{1}{\epsilon}f(s)(\|\nabla u\|^2_{L^2}+\|P(\rho)\|^2_{L^2})\|\rho(s,\cdot)\|^3_{L^\infty}\|\sqrt{\rho}u\|^8_{L^4}\\&+\epsilon f(s)(\|\sqrt{\rho}\partial_s u(s\cdot)\|^2_{L^2}+\|\rho(s,\cdot)\|_{L^\infty}\|g(s,\cdot)\|^2_{L^2})\big)+\frac{C}{\epsilon}f(s)\|k(\rho(s,\cdot)\|^2_{L^\infty}\\&\qquad\qquad\times\|\rho(s,\cdot)\|_{L^\infty}.\end{aligned}\tag{4.51}$$

From (4.42), we have:

$$\|\sqrt{\rho}u\|_{L^\infty_T(L^4)} \leq C\|\rho\|^{\frac{1}{4}}_{L^\infty_T(L^\infty)}(1+\|P(\rho)\|^2_{L^4(3}). \tag{4.52}$$

Therefore we have by using inequality (4.52), (4.45) and (4.51):

$$\begin{aligned} A(t) \leq C + C\int_0^t (\|h(\rho(s,\cdot))\|_{L^\infty} + \frac{1}{\epsilon}\|\rho(s,\cdot)\|^5_{L^\infty}\|\rho^{\frac{1}{4}}u\|_{L^4})f(s)\|\nabla u\|^2_{L^2} \\ + \frac{1}{\epsilon}f(s)\|P(\rho(s,\cdot))\|^2_{L^2}\|\rho(s,\cdot)\|^5_{L^\infty}\|\rho^{\frac{1}{4}}u\|_{L^4} + f(s)\|P(\rho(s,\cdot))\|^2_{L^\infty}\|\rho(s,\cdot)\|^{\frac{1}{2}}_{L^\infty} \\ \times|\rho^{\frac{1}{4}}u\|^2_{L^4} + \|\rho(s,\cdot)\|_{L^\infty}\|g(s,\cdot)^2_{L^2} + \phi(\|\rho(s,\cdot)\|_{L^\infty})ds. \end{aligned} \tag{4.53}$$

We have then by using (4.52)

$$\begin{aligned} A(t) &\leq C + C(1+(\int_0^t \|P(\rho(s,\cdot))\|^4_{L^\infty}ds)^4)\int_0^t \phi(\|\rho(s,\cdot)\|_{L^\infty})f(s)\big((1+\|\nabla u\|^2_{L^2} \\ &\qquad + \|P(\rho(s,\cdot))\|^2_{L^2)} + \|g(s,\cdot)\|^{2+\alpha}_{L^2}\big)ds, \\ &\leq C + C(1+\int_0^t \|P(\rho(s,\cdot))\|^{16}_{L^\infty}ds)\int_0^t \phi(\|\rho(s,\cdot)\|_{L^\infty})(A(s)+f(s)) + \|g(s,\cdot)\|^{2+\alpha}_{L^2})ds, \end{aligned} \tag{4.54}$$

where C depends of the time t here and $\alpha > 0$. Here ϕ is in $C^0(\mathbb{R}_+, \mathbb{R}^*_+) \cap C^1(0,\infty)$ such that $\phi(s) \leq M + Cs^\beta$ for some positive $M, C > 0$ and $\beta > 1$. We define by $\mathcal{F}$ this type of function. In the sequel as we will use a function ϕ_β with $\beta \in \mathbb{N}$ it will means that $\phi_\beta \in \mathcal{F}$. Gronwall's lemma provides the following bound:

$$\begin{aligned} A(t) \leq C(1+\int_0^t P^{16}(\|\rho(s,\cdot)\|_{L^\infty})ds\int_0^t \phi(\|\rho(s,\cdot)\|_{L^\infty}ds + \int_0^t \phi_1(\|\rho(s,\cdot)\|_{L^\infty}ds) \\ \times\exp(C(1+\int_0^t P^{16}(\|\rho(s,\cdot)\|_{L^\infty})ds)\int_0^t \phi_2(\|\rho(s,\cdot)\|_{L^\infty})ds)), \end{aligned} \tag{4.55}$$

where $\phi_1 \in C^0(\mathbb{R}_+, \mathbb{R}^*_+) \cap C^1(0,\infty)$ such that $\phi(s) \geq \epsilon_0 s$ for some positive s.
Next we have for $\phi_3 \in \mathcal{F}$ with α enough big:

$$\begin{aligned} \int_0^t P^4(\|\rho(s,\cdot)\|_{L^\infty})ds)\int_0^t \phi_2(\|\rho(s,\cdot)\|_{L^\infty})ds &\leq (\int \phi_3(\|\rho(s,\cdot)\|_{L^\infty})ds)^2, \\ &\leq \int \phi_3^2(\|\rho(s,\cdot)\|_{L^\infty})ds. \end{aligned}$$

Finally from (4.55), we obtain:

$$A(t) \leq C\exp(C\int_0^t \phi_4(\|\rho(s,\cdot)\|_{L^\infty})ds)). \tag{4.56}$$

Control of $\sup_{0<t\leq T} f^N(t)\int\rho|\dot{u}|^2(t,x)dx + \int\int f^N(s)|\nabla\dot{u}|^2dxds$

In the sequel, we want obtain estimate on ∇u in $L^1_T(BMO)$, that's why we need of additional regularity estimates. In this goal we follow the ideas of D. Hoff in [43]. We derive then estimates for the terms $f(t)^2\int_{\mathbb{T}^N}|\dot{u}|^2(t,x)dx$ and $\int_0^t\int_{\mathbb{T}^N} fN(s)|\nabla\dot{u}|^2dxds$. First we rewrite the momentum equation on the following form:

$$\rho\dot{u}-\mu\Delta u-(\lambda+\mu)\nabla\text{div}u+\nabla P(\rho)=\rho g.$$

We apply to the momentum equation the operator $\frac{d}{dt}=\partial_t+u\cdot\nabla$, we recall here the following equalities:

$$\begin{aligned}\frac{d}{dt}\rho\dot{u}^j&=\rho\frac{d}{dt}\dot{u}^j+\partial_t\rho\dot{u}^j+\dot{u}^j\sum_k\partial_k\rho\; u^k,\\&=\rho\frac{d}{dt}\dot{u}^j-\rho\text{div}u\dot{u}^j,\end{aligned}$$

We have next:

$$\begin{aligned}\mu\frac{d}{dt}\Delta u^j&=\mu\partial_t\Delta u^j+\sum_k\partial_k\Delta u^j u^k,\\&=\mu(\partial_t\Delta u^j+\text{div}(\Delta u^j u)-\Delta u^j\text{div}u),\end{aligned}$$

and (where $D=\text{div}u$):

$$(\lambda+\mu)\frac{d}{dt}\partial_j\text{div}u=(\lambda+\mu)(\partial_t\partial_jD+\text{div}(\partial_jDu)-\partial_jD\text{div}u),$$

We obtain finally:

$$\begin{aligned}\rho\frac{d}{dt}\dot{u}^j+\partial_j\partial_tP(\rho)+\text{div}\partial_jP(\rho)=\mu(\partial_t\Delta u^j+\text{div}(\Delta u^j u))\\+(\lambda+\mu)(\partial_t\partial_jD+\text{div}(\partial_jDu))+\rho\frac{d}{dt}g^j.\end{aligned}\tag{4.57}$$

We shall make use of the following transport theorem if $\rho\dot{w}=f_1$ and if $h=g(t)$, then:

$$\int_0^t\int_{\mathbb{T}^N}\frac{1}{2}\partial_s(h\rho w^2)dxds=\int_0^t\int_{\mathbb{T}^N}(\frac{1}{2}h'\rho w^2+hwf_1)dxds.$$

We apply the previous result with:

$f_1=-\partial_j\partial_tP(\rho)-\text{div}\partial_jP(\rho)+\mu(\partial_t\Delta u^j+\text{div}(\Delta u^ju)+(\lambda+\mu)(\partial_t\partial_jD+\text{div}(\partial_jDu))+\rho\dot{g}$,

and with $h(s)=f(s)^N$, we obtain then by summing over j:

$$\begin{aligned}\frac{1}{2}f(t)^N\int_{\mathbb{T}^N}\rho(t,x)|\dot{u}(t,x)|^2dx=\int_0^t\int_{\mathbb{T}^N}Nf^{N-1}(s)f'(s)\rho|\dot{u}|^2dxds\\+\int_0^t\int_{\mathbb{T}^N}f(s)^N\dot{u}^j\big[-(\partial_j\partial_tP+\text{div}(\partial_jPu)+\mu[\Delta\partial_tu^j+\text{div}(\Delta u^ju)]\\+(\lambda+\mu)[\partial_j\partial_tD+\text{div}(\partial_jDu)]+\rho\dot{g}^j\big]dxds.\end{aligned}\tag{4.58}$$

Since $f^{N-1}(s)f'(s)\leq f(s)$ we can apply (4.56) to bound the first term on the right. Next by integrations by part we get:

$$\begin{aligned}-\int_0^t\int_{\mathbb{T}^N}f(s)^N\dot{u}^j(\partial_j\partial_tP+\text{div}(\partial_jPu))dsdx=\int_0^t\int_{\mathbb{T}^N}f(s)^N(\partial_j\dot{u}^j\partial_tP+\partial_k\dot{u}^j\partial_jPu^kdsdx,\\=\int\int f(s)^NP'(\partial_j\dot{u}^j\partial_t\rho+\partial_k\dot{u}^j\partial_j\rho u^k)dsdx\end{aligned}$$

$$\int_0^t \int_{\mathbb{T}^N} f(s)^N P^{'}[-\partial_j \dot{u}^j(\rho\partial_k u^k + \partial_k \rho u^k) + \partial_k \dot{u}^j \partial_j \rho u^k]dxds,$$
$$= -\int_0^t \int_{\mathbb{T}^N} f(s)^N [P^{'}\rho D\partial_j \dot{u}^j + \partial_k P u^k \partial_j \dot{u}^j - \partial_j P u^k \partial_k \dot{u}^j]dxds,$$
$$= -\int_0^t \int_{\mathbb{T}^N} f(s)^N [P^{'}\rho D\partial_j \dot{u}^j - P(D\partial_j \dot{u}^j - \partial_j u^k \partial_k \dot{u}^j)]dxds.$$

This term is therefore bounded in absolute value by:

$$C\big(\int_0^t \int_{\mathbb{T}^N} P(\rho) f(s)^N |\nabla u|^2 dxds\big)^{\frac{1}{2}} \big(\int_0^t \int_{\mathbb{T}^N} f(s)^N |\nabla \dot{u}|^2 dxds\big)^{\frac{1}{2}}$$
$$\leq C_\epsilon \int_0^t \int_{\mathbb{T}^N} P(\rho) f(s)^N |\nabla u|^2(s,x) dxds + \epsilon \int_0^t \int_{\mathbb{T}^N} f(s)^N |\nabla \dot{u}(s,x)|^2 dxds.$$

The third term on the right side of (4.58) may be written:

$$-\mu \int_0^t \int_{\mathbb{T}^N} f^N [\nabla \dot{u}^j \cdot \nabla u^j_t + (\nabla \dot{u}^j \cdot u)\Delta u^j]dxds$$
$$= -\mu \int_0^t \int_{\mathbb{T}^N} f^N [\nabla \dot{u}^j \cdot (\nabla u^j_t + \nabla(\nabla \dot{u}^j \cdot u)) + \dot{u}^j_k (u^k u^j_{ll} - (u^j_l u^l)_k)]dxds,$$
$$= -\mu \int_0^t \int_{\mathbb{T}^N} f^N [\nabla \dot{u}^j \cdot (\nabla u^j_t + \nabla(\nabla \dot{u}^j \cdot u)) + \dot{u}^j_k (u^k u^j_{ll} - u^j_{lk} u^l - u^j_l u^l_k)]dxds,$$
$$\leq -\mu \int_0^t \int_{\mathbb{T}^N} f^N |\nabla \dot{u}^j|^2 + M \int\int f^2 |\nabla u|^2 (|\nabla \dot{u}| + |\dot{D}|)dxds.$$

The last term on the right side of (4.58) may be bound as follows:

$$-(\lambda+\mu)\int_0^t \int_{\mathbb{T}^N} f^N(\partial_j \partial_t D + \mathrm{div}(\partial_j D u)dxds \leq -(\lambda+\mu)\int_0^t \int_{\mathbb{T}^N} f^N \dot{D}^2 dxds$$
$$+ M \int_0^t \int_{\mathbb{T}^N} f^N |\nabla u|^2 (|\nabla \dot{u}| + |\dot{D}|)dxds.$$

It then follows by Young's inequalities that:

$$\begin{aligned}
&f(t)^N \int_{\mathbb{T}^N} \rho |\dot{u}|^2(t,x)dx + \int_0^t \int_{\mathbb{T}^N} f^N(s)(\mu|\nabla \dot{u}|^2 + (\lambda+\mu)|\dot{D}|^2)dxds \\
&\leq M\big[C_0 + C_\epsilon C_0 \|P(\rho)\|_{L^\infty} + \int_0^t \int_{\mathbb{T}^N} f(s)\rho|\dot{u}|^2 dxds \\
&\qquad + \int_0^t \int_{\mathbb{T}^N} f^N(s)|\nabla u|^4 dxds] + + \int_0^t \int_{\mathbb{T}^N} f^N(s)\rho|\dot{g}|^2 dxds, \\
&\leq M\big[C_0 + C_\epsilon C_0 \|P(\rho)\|_{L^\infty} + A(t) + \int_0^t \int_{\mathbb{T}^N} f^N(s)|\nabla u|^4 dxds \\
&\qquad + \int_0^t \int_{\mathbb{T}^N} f^N(s)\rho|\dot{g}|^2 dxds].
\end{aligned} \tag{4.59}$$

In the sequel we recall that $\omega = \mathrm{curl} u$. Next from the momentum equation, we obtain as in the works of D. Hoff in [43]:

$$\mu|\nabla\omega|^2 = \mu(\mathrm{div}(\omega\nabla\omega) + \partial_j(\rho\omega\dot{u}^k) - \partial_k(\rho w\dot{u}^j) + \rho(\dot{u}^j\partial_k\omega - \dot{u}^k\partial_j\omega).$$

Integrating we thus obtain:

$$\begin{aligned}\int_0^t\int_{\mathbb{T}^N} f(s)|\nabla\omega|^2 dxds &\leq M(\int_0^t\int_{\mathbb{T}^N} f(s)\rho^2|\dot{u}|^2(s,x)dsdx\\ &\leq M\|\rho\|_{L^\infty}A(t).\end{aligned} \tag{4.60}$$

Similarly we obtain:

$$\begin{aligned}\sup_{0<t\leq T} f(t)^N\int|\nabla\omega|^2dx &\leq M(\int_{\mathbb{T}^N} f^N(t)\rho|\dot{u}|^2(t,x)dsdx + \int_{\mathbb{T}^N} f(t)^N\rho|w|^2(t,x)dx)\\ &\leq M(\int_{\mathbb{T}^N} f^N(t)\rho|\dot{u}|^2(t,x)dsdx + C\|\rho\|_{L^\infty}),\\ \leq M(C_0 + C_\epsilon C_0\|P(\rho)\|_{L^\infty} &+ A(t) + \int_0^t\int_{\mathbb{T}^N} f^N(s)|\nabla u|^4 dxds + C\|\rho\|_{L^\infty}).\end{aligned} \tag{4.61}$$

To complete the estimates (4.59) and (4.61), we will need of estimation on the term $\int_0^t\int_{\mathbb{T}^N} f^N(s)|\nabla u|(s,x)^4 dxdt$ and to conclude we will use (4.56).

Control of $\int_0^t\int_{\mathbb{T}^N} f^N(s)|\nabla u|(s,x)^4 dxdt$

We have then:

$$\int_0^t\int_{\mathbb{T}^N} f^N|\nabla u|^4 dsdx \leq \int_0^t\int_{\mathbb{T}^N} f^N\big((RG)^4+\omega^4)(s,x)+f^N(s)RP(\rho)^4(s,x)\big)dxds. \tag{4.62}$$

Let focus us on the case $N=3$. When $N=3$ we can apply Gagliardo-Nirenberg, we have then:

$$\begin{aligned}\int_0^t\int_{\mathbb{T}^N} f^3(s)(RG)^4(s,x)dsdx &\leq \int_0^t f(s)^3(\int_{\mathbb{T}^N} G^2dx)^{\frac{1}{2}}(\int_{\mathbb{T}^N}|\nabla G|^2dx)^{\frac{3}{2}}dt,\\ &\leq \|\sqrt{f(t)}G\|_{L_t^\infty(L^2)}\|f(t)^{\frac{3}{2}}\nabla G\|_{L_t^\infty(L^2)}\int_0^t\int_{\mathbb{T}^N} f(s)|\nabla G|(s,x)^2 dsdx.\end{aligned} \tag{4.63}$$

By the definition of G we obtain:

$$f(t)\int_{T^N} G^2(t,x)dx \leq M[\|P(\rho)\|_{L^\infty(L^2)} + A(t)].$$

Moreover as $(\lambda+2\mu)\Delta G = \mathrm{div}(\rho(\dot{u}+g))$, we have:

$$f(t)^3\int_{\mathbb{T}^N}|\nabla G|^2dx \leq Mf(t)^3\|\rho\|_{L^\infty}^{\frac{1}{2}}(\int_{\mathbb{T}^N}\rho|\dot{u}|^2(t,x)dx + \|\rho\|_{L^\infty}^{\frac{1}{2}}\int_{\mathbb{T}^N}|g|^2(t,x)dx). \tag{4.64}$$

We obtain finally by using (4.63) and (4.64):

$$\int_0^t\int_{\mathbb{T}^N} f^3(s)(RG)^4(s,x)dsdx \leq M\|\rho\|_{L^\infty}^\alpha(1+A(t)^2) \tag{4.65}$$

with $\alpha > 0$.
A similar argument applies to the vorticity term, so that:

$$\int_0^t \int_{\mathbb{T}^N} f^3(s)|\nabla u|^4(s,x)dsdx \le M\|\rho\|^{\alpha}_{L^\infty}(1 + A^2(t) + \|g\|_{L^\infty(L^2)}+) \tag{4.66}$$

From (4.56) and (4.59), (4.61), we can conclude that:

$$\begin{aligned} B(t)f(t)^2\int_{\mathbb{T}^N}\rho|\dot{u}|^2(t,x)dx + \int_0^t\int_{\mathbb{T}^N} f^2(s)(\mu|\nabla\dot{u}|^2 + (\lambda+\mu)|\dot{D}|^2)dxds \\ \le M\big[C_0 + C_\epsilon C_0\|P(\rho)\|_{L^\infty} + CC_e A(t)(1 + A(t)^9)\big]. \end{aligned} \tag{4.67}$$

$$\begin{aligned} \sup_{0<t\le T} f(t)^2\int|\nabla\omega|^2dx \le M(C_0 + C_\epsilon C_0\|P(\rho)\|_{L^\infty} + A(t)C\|\rho\|_{L^\infty} \\ + \epsilon B(t) + CC_e A(t)^{10}). \end{aligned} \tag{4.68}$$

We can remark that all the inequalities (4.56), (4.60), (4.67) and (4.68) depend on the control of $\|\rho\|_{L^\infty}$. In the following subsection, we want explain that we can control ρ in L^∞ in finite time.

Conclusion

We will treat by simplicity only the case $N = 3$. We want here to explain how to control the norm L^∞ of the density ρ in finite time. From (3.36), we have:

$$\begin{aligned} \log(\rho(t,x)) \le & \log(\|\rho_0\|_{L^\infty}) + C\|(\Delta)^{-1}\text{div}m_0\|_{L^\infty} + C\|(\Delta)^{-1}\text{div}(\rho u)\|_{L^\infty} \\ & + C\int_0^t \|[u_j, R_iR_j](\rho u_i)(s,\cdot)\|_{B^1_{N+\epsilon,1}}ds, \end{aligned}$$

We have then by Sobolev embedding with $\epsilon > 0$, (4.42) and let $p \ge 6+\epsilon$ with $\epsilon > 0$ and $q = \frac{\gamma p}{p-1}$ such that $\frac{1}{p} + \frac{1}{q} < \frac{1}{3}$ which means $\gamma \ge 6$:

$$\begin{aligned} \|(\Delta)^{-1}\text{div}(\rho u)\|_{L^\infty_T(L^\infty)} \le & \|\rho u\|_{L^\infty_T(L^{3+\epsilon})}, \\ & \le \|\rho\|^{\frac{p-1}{p}}_{L^\infty_T(L^\gamma)}\|\rho^{\frac{1}{p}}u\|_{L^\infty_T(L^p)} \\ & \le C\|\rho\|^{\frac{p-1}{p}}_{L^\infty_T(L^\gamma)}(1 + \int_0^t \|P(\rho(s,\cdot))\|^{6+\epsilon}_{L^\infty}ds). \end{aligned} \tag{4.69}$$

We can proceed similarly in the case $N = 2$ and $\gamma \ge 1$ suffices.
We want show now that $\dot{u} \in L^1(L^{2-\epsilon})$, we have early:

$$\begin{aligned} \|\nabla u\|_{L^6} \le & \|G\|_{L^6} + \|P(\rho)\|_{L^\infty} + \|\omega|_{L^6} \\ & \le \|\rho\|^{\frac{1}{2}}_{L^\infty}\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2} + \|\rho\|_{L^\infty}\|g\|_{L^2} + \|P(\rho)\|_{L^\infty} + \|\nabla w\|_{L^2}. \end{aligned}$$

We have then by interpolation with θ small:

$$\begin{aligned} f(t)^{\frac{1-\theta}{2}}\|\nabla u\|_{L^{6-\alpha}} \le \|\nabla u\|^\theta_{L^2}(\|\rho\|^{\frac{1}{2}}_{L^\infty}\sqrt{f(t)}\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2} + \|\rho\|_{L^\infty}\|g\|_{L^2} \\ + \|P(\rho)\|_{L^\infty} + \sqrt{f(t)}\|\nabla\omega\|_{L^2})^{1-\theta} \end{aligned}$$

We obtain then:

$$\int_0^t \|[u_j, R_iR_j](\rho u_i)(s,\cdot)\|_{B^1_{N+\epsilon,1}} ds \leq \int_0^t \|\rho u(s,\cdot)\|_{L^{6+\epsilon}} \|\nabla u(s,\cdot)\|_{L^{6-\alpha}} ds$$

$$\leq \int_0^t \|\rho u(s,\cdot)\|_{L^{6+\epsilon}} \|\nabla u\|_{L^2}^{\theta} (\|\rho\|_{L^\infty}^{\frac{1}{2}} \sqrt{f(t)} \|\rho^{\frac{1}{2}}\dot{u}\|_{L^2} + \|\rho\|_{L^\infty}\|g\|_{L^2}$$

$$+ \|P(\rho)\|_{L^\infty} + \sqrt{f(t)}\|\nabla\omega\|_{L^2})^{1-\theta} dx$$

$$\leq \|\rho^{\frac{1}{6+\epsilon}} u(s,\cdot)\|_{L^\infty_t(L^{6+\epsilon})} \int_0^t \|\rho^{1+\frac{1-\theta}{2}-\frac{1}{6}}\|_{L^\infty} \|\nabla u\|_{L^2}^{\theta} (\sqrt{f(t)}\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2} + \|\rho\|_{L^\infty}\|g\|_{L^2}$$

$$+ \|P(\rho)\|_{L^\infty} + \sqrt{f(t)}\|\nabla\omega\|_{L^2})^{1-\theta} dx$$

$$\leq C\int_0^t \|\nabla u(,\cdot)\|_{L^2}^{\frac{2\theta}{1-\theta}} \|\rho(s,\cdot)\|_{L^\infty}^{\beta} ds + A(t) + \int_0^t \phi(\|\rho(s,\cdot)\|_{L^\infty}) ds, \leq C + A(t) + \int_0^t \phi(\|\rho(s,\cdot)\|_{L^\infty}) ds.$$

By using the previous inequality and (3.36), (4.56) and (4.69) we conclude that:

$$\log\rho(t,x) \leq C + \int_0^t \phi(\|\rho(s)\|_{L^\infty}) ds + A(t),$$
$$\leq C_t + C\exp(\int_0^t \phi_1(\|\rho(s,\cdot)\|_{L^\infty}) ds)).$$

with $\phi \in \mathcal{F}$. We obtain then:

$$|\rho(t,x)| \leq C\exp(\int_0^t \phi_1(\|\rho(s)\|_{L^\infty}) ds). \tag{4.70}$$

Denoting by $w(t)$ the right-hand side of (4.70), we conclude that:

$$\frac{d}{dt} w(t) \leq C\phi_2(\|\rho(s)\|_{L^\infty}) w(t) \leq C w(t)(M + C_1 w^\beta(t)),$$

this because $\phi_2 \in \mathcal{F}$. We have then as $w(t) \geq C$:

$$\frac{\frac{d}{dt}w(t)}{w^{1+\beta}(t)} \leq M_1$$

so that there exists T_0 such that for all $T < T_0$

$$\|\rho\|_{L^\infty((0,T)\times\mathbb{T}^N)} \leq C_T.$$

Proof of (1.13) and (1.15)

We want get now solutions such that we control ∇u in $L^1(BMO)$. To do it, we need of additional regularity on the velocity. We will use again the technics introduced by D. Hoff in [39]. The idea is to obtain some estimates by interpolation in "killing" the coupling between pressure and velocity. First , we mollify initial data satisfying the conditions of theorem 1.2 and then appeal to the result of [36] to otain a solution (ρ, u) defined at least for small time. We want now derive estimates on the solution independent of the mollifier and depending only on the initial data.

In the sequel we will treat only by simplicity the case $N = 3$. Fixing the local in time solution (ρ, u) described above on the interval $[0, T_0]$ with $T_0 > 0$, we therefore assume throughout this section that $C^{-1} \leq \rho \leq C$ with $C > 0$. We define a differential operator $\mathcal{L}$ acting on functions $w : [0, T] \times \mathbb{T}^N \to \mathbb{T}^N$ by

$$\mathcal{L}w = \partial_t(\rho w) + \mathrm{div}(\rho u \otimes w) - \mu \Delta w - \lambda \nabla \mathrm{div} w,$$

and we define w_1 and w_2 by:

$$\mathcal{L}w_1 = 0, \;\; (w_1)_{/t=0} = u_0, \quad \mathcal{L}w_2 = -\nabla P(\rho), \;\; (w_2)_{/t=0} = 0. \tag{4.71}$$

We observe here that by uniqueness $w_1 + w_2 = u$. Straightforward energy estimates then show that:

$$\sup_{0 \leq t \leq T_0} \int_{\mathbb{T}^N} |w_1(t,x)|^2 dx + \int_0^{T_0} \int_{\mathbb{T}^N} |\nabla w_1|^2 dx dt \leq C \int_{\mathbb{T}^N} |u_0|^2 dx \tag{4.72}$$

and:

$$\sup_{0 \leq t \leq T_0} \int_{\mathbb{T}^N} |w_2(t,x)|^2 dx + \int_0^{T_0} \int_{\mathbb{T}^N} |\nabla w_2|^2 dx dt \leq CT \sup_{0 \leq t} |P(\rho(t,\cdot)|^2. \tag{4.73}$$

We shall derive (1.13) and (1.15) simultaneously as consequences of estimates for the following quantities:

$$\sup_{0 \leq t \leq T_0} t^{1-k} \int_{\mathbb{T}^N} |\nabla w_1(t,x)|^2 dx + \int_0^{T_0} \int_{\mathbb{T}^N} t^{1-k} |\dot{w}_1|^2 dx dt,$$

for $k = 0, 1$ and,

$$\sup_{0 \leq t \leq 1} \int |\nabla w_2(t,x)|^2 dx + \int_0^1 \int |\dot{w}_2|^2 dx dt.$$

To derive these bounds, we multiply equation (4.71) for w_1 and w_2 by $\dot{w}_1$ and $\dot{w}_2$, respectively and integrate. The details which are nearly identical to those in the previous section are quite straightforward, the essential point being that the spatial gradient of ρ must be avoided. Indeed the procedure of D. Hoff in [39] allows to "kill" the coupling between the velocity and the pressure. The results are that with $C' > 0$:

$$\begin{aligned} &\frac{1}{2}\mu (\tau^k \int |\nabla w_1(\tau,x)|^2 dx)\big|_{\tau=0}^{\tau=t} + \int_0^t \int_{\mathbb{T}^N} \tau^k \rho |\dot{w}_1|^2 dx d\tau \leq \\ &\qquad \frac{1}{2}\mu k \int_0^t \int_{\mathbb{T}^N} \tau^{k-1} |\nabla w_1|^2 dx d\tau + \int_0^t \int_{\mathbb{T}^N} \tau^k |\nabla w_1|^2 |\nabla u| dx d\tau, \end{aligned} \tag{4.74}$$

and

$$\begin{aligned} &\frac{1}{2}\mu \int_{\mathbb{T}^N} |\nabla w_2(\tau,x)|^2 dx + \int_0^t \int_{\mathbb{T}^N} \rho |\dot{w}_2|^2 dx d\tau \leq \\ &\qquad \int_{\mathbb{T}^N} P(\rho(t,\cdot)) \mathrm{div} w_2(\cdot,x) dx\big|_0^t + \int_0^t \int_{\mathbb{T}^N} (|\nabla w_2|^2 |\nabla u| + |\nabla w_2||\nabla u|) dx d\tau, \end{aligned} \tag{4.75}$$

By proceding exactly as in the previous section, we obtain the following results:

$$\sup_{0\le t\le T_0}\int_{\mathbb{T}^N}|\nabla w_1(t,x)|^2dx+\int_0^{T_0}\int_{\mathbb{T}^N}|\dot{w}_1|^2dxdt\le C\|u_0\|_{H^1}^2, \tag{4.76}$$

$$\sup_{0\le t\le T_0} t\int_{\mathbb{T}^N}|\nabla w_1(t,x)|^2dx+\int_0^{T_0}\int t|\dot{w}_1|^2dxdt\le C\|u_0\|_{L^2}^2, \tag{4.77}$$

$$\sup_{0\le t\le T_0}\int_{\mathbb{T}^N}|\nabla w_2(t,x)|^2dx+\int_0^{T_0}\int_{\mathbb{T}^N}|\dot{w}_2|^2dxdt\le CC_0, \tag{4.78}$$

where C_0 depends only of the initial data. Now since the solution operator $u_0\longrightarrow w_1(t,\cdot)$ is linear, we can apply a standard Riesz-Thorin interpolation argument to deduce from (4.76) and (4.77) that:

$$\sup_{0\le t\le T_0} t^{1-\beta}\int^{\mathbb{T}^N}|\nabla w_1(t,x)|^2dx+\int_0^{T_0}\int_{\mathbb{T}^N}t^{1-\beta}|\dot{w}_1|^2dxdt\le C\|u_0\|_{H^\beta}^2. \tag{4.79}$$

As $u=w_1+w_2$, we then conclude from (4.78) and (4.79) that:

$$\sup_{0\le t\le T_0} t^{1-\beta}\int_{\mathbb{T}^N}|\nabla u(t,x)|^2dx+\int_0^{T_0}\int_{\mathbb{T}^N}t^{1-\beta}|\dot{u}|^2dxdt\le CC_0\|u_0\|_{H^\beta}^2. \tag{4.80}$$

The next step is to obtain bounds for the terms

$$\sup_{0\le t\le T_0} t^{2-\beta}\int_{\mathbb{T}^N}|\nabla u(t,x)|^2dx+\int_0^{T_0}\int_{\mathbb{T}^N}t^{2-\beta}|\nabla\dot{u}|^2dxdt$$

appearing in (1.13). To do this, we multiply the momentum equation of (1.1) by $t^{2-\beta}\dot{u}$ and integrate. The details are exactly as in the previous section, except now we apply the β dependent smoothing rates established in (4.79). Combining these bounds with (4.72), (4.73) and (4.79), we then obtain (1.13) for times $t\le T_0$. To prove (1.15), we observe that for $k=0,1$,

$$\sup_{0\le t\le 1}\|w_1(t,\cdot)\|_{H^k}\le C\|u_0\|_{H^k},$$

by (4.72) and (4.76). Thus:

$$\sup_{0\le t\le 1}\|w_1(t,\cdot)\|_{H^\beta}\le C\|u_0\|_{H^k},$$

for $\beta\in[0,1]$. As $u=w_1+w_2$, and applying (4.78) we obtain that:

$$\sup_{0\le t\le 1}\|w_1(t,\cdot)\|_{H^\beta}\le CC_0,$$

and then for $r\in(2,\frac{6}{3-2\beta})$ in the case that $\beta>0$, that:

$$\sup_{0\le t\le 1}\|u(r,\cdot)-\widetilde{u}\|_{L^r}\le CC_0.$$

This proves (1.15).

Regularity on the gradient of the velocity u in $L^1_{T_0}(W^{1,\alpha}+BMO)$ with $\alpha>N$

Here we want examine the regularity of the gradient of the velocity and prove that ∇u is in $L^1_{T_0}(BMO)$ to prove (1.16). More precisely we will verify that the new variable v_1 introduced in [36] called "effective velocity' belongs in $L^1_{T_0}(W^{2,\alpha})$ with $\alpha>N$, let $\nabla v_1\in L^1T_0(L^\infty)$. We recall here the definition of v_1 introduced in [36].The idea of [36] was to introduce a variable v_1 which allows to kill the coupling between the velocity and the pressure in the momentum equation of (1.1).In this goal, we need to integrate the pressure term in the study of the linearized equation of the momentum equation. To do this, we will try to express the gradient of the pressure as a Laplacian term, so we set:

$$\Delta v=\nabla P(\rho).$$

We have then $v=(\Delta)^{-1}\nabla P(\rho)$ with $(\Delta)^{-1}$ the inverse Laplacian with zero value on $\mathbb{T}^N$. In the sequel we will set:

$$v_1=u-\frac{\lambda+2\mu}{v}.$$

We have then:

$$\begin{aligned}\Delta u=&\nabla F+\text{div}\omega+(2\mu+\lambda)^{-1}\nabla(P(\rho)),\\ =&\Delta v_1+(2\mu+\lambda)^{-1}\nabla(P(\rho)).\end{aligned}\tag{4.81}$$

We can easily show that $\int_0^{T_0}\|\nabla v\|_{L^\infty}dt<+\infty$ if (1.13) holds.To see this, we apply standard elliptic theory on v_1 and the fact that $\Delta G=\text{div}(\rho\dot{u}-\rho g)$. We will use in particular the fact that $\text{div}v_1=(\lambda+2\mu)G$ and $\text{curl}v_1=\omega$.We consider here only the case $N=3$. The case $N=2$ follows the same lines. For some $\alpha>3$ and $\epsilon>0$ determined by α we have then:

$$\begin{aligned}\|\nabla v_1\|_{L^\infty}&\leq C(\|\nabla F\|_{L^\alpha}+\|\nabla\omega\|_{L^\alpha}),\\ &\leq C(\|\rho\dot{u}(t,\cdot)\|_{L^\alpha}+\|\rho g\|_{L^\alpha}+\|\nabla\omega\|_{L^\alpha}),\\ &\leq C(\|\rho\|_{L^\infty}\|\dot{u}(t,\cdot)-\widetilde{\dot{u}}\|_{L^2}^{\frac{1-\epsilon}{2}}\|\nabla\dot{u}(t,\cdot)\|_{L^2}^{\frac{1+\epsilon}{2}}+\|\rho\|_{L^\infty}\|g\|_{L^\alpha}+\|\nabla\omega\|_{L^\alpha}),\\ &\leq C(\|\rho\|_{L^\infty}(\|\dot{u}(t,\cdot)\|_{L^2}^{\frac{1-\epsilon}{2}}+\widetilde{\dot{u}}^{\frac{1-\epsilon}{2}})\|\nabla\dot{u}(t,\cdot)\|_{L^2}^{\frac{1+\epsilon}{2}}+\|\rho\|_{L^\infty}\|g\|_{L^\alpha}+\|\nabla\omega\|_{L^\alpha}),\end{aligned}$$

We recall here that $\widetilde{\rho\dot{u}}=\widetilde{\rho g}$, we have then as $\frac{1}{\rho}\geq C$ on $[0,T_0]$:

$$\widetilde{\dot{u}}\leq\frac{1}{\|\rho\|_{L^\infty_{T_0}}}\widetilde{\rho g}.$$

so that by using (1.13), we obtain:

$$\begin{aligned}\int_0^{T_0}\|\nabla v_1\|_{L^\infty}dt&\leq C\int_0^{T_0}t^\beta(t^{1-s}\int_{\mathbb{T}^N}|\dot{u}|^2dx)^{\frac{1-\epsilon}{4}}(t^\sigma\int_{\mathbb{T}^N}|\nabla\dot{u}|^2dx)^{\frac{1+\epsilon}{4}}dt+C_0,\\ &\leq C(\int_0^{T_0}t^{2\beta}dt)^{\frac{1}{2}}+C_0,\end{aligned}$$

with $s=\frac{N}{2}+\epsilon-1$ $(\epsilon>0)$ and where $4\beta=(s-1)(1-\epsilon)-(\sigma+\epsilon)$. The above integral is therefore finite as $2\beta>-1$. A similar result result holds for $N=2$ with $s>0$. Thus

for the solution constructed in the previous section, $\int_0^T \|\nabla v_1(t,\cdot)\|_{L^\infty} dt$ is finite if (1.13) holds, $\inf \rho_0 \geq c > 0$ and $u_0 \in H^{\frac{N}{2}+\epsilon-1}$,with $\epsilon > 0$.
More precisely we have proved in fact that:

$$\nabla v_1 \in L^1_T(W^{1,\alpha}) \hookrightarrow L^1_T(B^{1+\epsilon}_{N,1}). \tag{4.82}$$

with $\alpha = N + 2\epsilon$ where $\epsilon > 0$.
As $P(\rho) \in L^\infty$ we deduce from (4.81) and the results of Calderon-Zygmund, that:

$$\nabla u \in L^1_{T_0}(BMO).$$

5 Proof of theorem 1.2

The above arguments are not rigorous, since we have to assume that $\rho(t,x)$ is positive for all (t,x). In order to deal with possibly vanishing densities, we remark that if we assume as in [36] that ρ_0 is also bounded from below, we can get L^∞ bounds for $\log \rho$. In that case, when $N = 2, 3$, there exists $T_0 > 0$ such that for all $t < T_0$:

$$\|\rho\|_{L^\infty((0,t)\times\mathbb{T}^N)} + \|\frac{1}{\rho}\|_{L^\infty((0,t)\times\mathbb{T}^N)} \leq C_t. \tag{5.83}$$

Thus ρ is also bounded from below for small enough times, so that vacuum does not form on $[0, T_0]$. It follows that starting from a general bounded initial density ρ_0, we can apply the above arguments to a weak solution (ρ^n, u^n) corresponding to initial values $\rho_0^n = \rho_0 + \frac{1}{n}$ and $u_0^n = u_0$ which converge strongly in $L^\infty(\mathbb{T}^N)$ to ρ_0 and u_0. In view of the weak stability results of system (1.1) given by E. Feireisl et al in [30], ρ^n and $\sqrt{\rho^n} u^n$ respectively converge to ρ and u in $C([0,T_0], L^q(\mathbb{T}^N))$ and $L^2((0,T_0)\times\mathbb{T}^N)$ for all $q < \gamma - 1 + \frac{2\gamma}{N}$. Hence the uniform L^∞ bounds on ρ^n yield L^∞ bounds on ρ.
In the above formal derivations, we assumed that there exists global weak solutions of (1.1). This problem does not occur when we take γ larger than $\frac{N}{2}$ via the works of Feireisl et al in [30]. When $1 < \gamma \leq \frac{N}{2}$, we can approximate solutions of (1.1) by a global weak solutions of (1.1) by a global weak solution (ρ^n, u^n) corresponding to a modified pressure law that satisfies:

$$P_n(\rho) = P(\rho) + \frac{1}{n}\rho^2, \tag{5.84}$$

and the same initial data (ρ_0, u_0). Applying the above arguments on (ρ^n, u^n), we obtain all the uniform bounds for ρ^n and u^n on $[0, T_0]$, where T_0 does not depend on n.As a result, we also have uniform $L^2((0,T_0)\times\mathbb{T}^N)$ bounds on ∇u^n and $L^\infty(0,T_0,L^2(\mathbb{T}^N))$ bounds on $\sqrt{\rho^n} u^n$. Hence the weak stability results hold since the initial data are not functions of n and (ρ^n, u^n) converge to (ρ, u) in $L^2((0,T_0)\times\mathbb{T}^N)^{N+1}$, where (ρ, u) is a solution of (1.1) on $(0, T_0)$. We refer to [50] for complete details of the stability proof. Let us emphasize that one of the key arguments for proving weak stability of solutions of (1.1) is to obtain uniform $L^2((0,T_0)\times\mathbb{T}^N)$ bounds on ρ^n to renormalize the transport equation. This is the case here as we have uniforms bounds on the density in $L^\infty((0,T_0)\times\mathbb{T}^N)$.

6 Proof of corollary 1

6.0.1 Control of $\rho \in L^\infty(B^\epsilon_{\infty,\infty})$

In view of proposition 2.6 where in our case $h(\rho) = P(\rho)$, we have for all $t \in [0, T]$ and $0 < \epsilon < 1$:

$$\|\rho\|_{\widetilde{L}^\infty_t(B^\epsilon_{\infty,\infty})} \leq e^{CV(t)}(1 + \|\rho_0\|_{B^\epsilon_{\infty,\infty}}), \tag{6.85}$$

where $V(t) = \int_0^t \big(\|\nabla u(\tau)\|_{L^\infty} + \|\text{div}v_1(\tau)\|_{B^\epsilon_{\infty,\infty}} + \|\rho(\tau)\|^s_{L^\infty}\big)d\tau$, where s the smallest integer such that $P' \in W^{s,\infty}$. We have seen by (4.82) that $\nabla v_1 \in L^1(0, T, B^\epsilon_{\infty,\infty})$ with $\epsilon > 0$, the main difficulty is to control $\nabla u \in L^1(0, T, L^\infty)$, for this we recall that:

$$\begin{aligned}\|\nabla u\|_{L^1_T(L^\infty)} &\leq \|\nabla v_1\|_{L^1_T(B^\epsilon_{\infty,\infty})} + \|P(\rho)\|_{L^1_T(B^0_{\infty,1})},\\ &\leq \|\nabla v_1\|_{L^1_T(B^\epsilon_{\infty,\infty})} + \|\rho\|^s_{L^\infty_T(L^\infty)}\|\rho\|_{L^1_T(B^0_{\infty,1})}.\end{aligned}$$

Unsurprisingly the result comes from the well known following estimate: Next we have:

$$\|\rho(t)\|_{B^0_{\infty,1}} \leq C\|\rho(t)\|_{B^0_{\infty,\infty}} \log(e + \frac{\|\rho(t)\|_{B^\epsilon_{\infty,\infty}}}{\|\rho(t)\|_{B^0_{\infty,\infty}}}),$$

and we recall the following inequality:

$$\forall x > 0,\ \forall \delta > 0,\ \log(e + \frac{\delta}{x}) \leq \log(e + \frac{1}{x})(1 + \log \delta).$$

We obtain then from the previous inequality

$$\|\rho(t)\|_{B^0_{\infty,1}} \leq \|\rho(t)\|_{B^0_{\infty,\infty}}\big(1 + \log(\|\rho(t)\|_{B^\epsilon_{\infty,\infty}})\big) \log(e + \frac{1}{\|\rho(t)\|_{B^0_{\infty,\infty}}}),$$

Let $X(t) = \int_0^t \|\rho(s)\|_{B^0_{\infty,1}} ds$, we have then:

$$V(t) \leq C\big(X(t) + \int_0^t \big(\|\nabla v_1(\tau)\|_{L^\infty} + \|\text{div}v_1(\tau)\|_{B^\epsilon_{\infty,\infty}})d\tau\big).$$

Combining (6.85) and the previous inequality leads to:

$$\begin{aligned}X(t) &\leq \int_0^t \|\rho(s)\|_{B^0_{\infty,\infty}}\big(1 + CV(t) + \log(1 + \|\rho_0\|_{B^\epsilon_{\infty,\infty}})\big) \log(e + \frac{1}{\|\rho(s)\|_{B^0_{\infty,\infty}}})ds,\\ &\leq \int_0^t \|\rho(s)\|_{B^0_{\infty,\infty}}\big(1 + CX(t) + C\int_0^t \big(\|\nabla v_1(\tau)\|_{L^\infty} + \|\text{div}v_1(\tau)\|_{B^\epsilon_{\infty,\infty}}\big)d\tau\\ &\qquad + \log(1 + \|\rho_0\|_{B^\epsilon_{\infty,\infty}})\big) \log(e + \frac{1}{\|\rho(s)\|_{B^0_{\infty,\infty}}})ds.\end{aligned}$$

Applying Gronwall inequality and inequality (4.82) shows that:

$$\begin{aligned}X(t) &\leq C_{t,0} \exp(C\int_0^t \|\rho(s)\|_{B^0_{\infty,\infty}} \log(e + \frac{1}{\|\rho(s)\|_{B^0_{\infty,\infty}}})ds),\\ &\leq C_{t,0} \exp(C\int_0^t (1 + \|\rho(s)\|_{L^\infty})ds),\end{aligned}$$

where $C_{t,0}$ depnds only of the time t and the initial data. As $\rho \in L^\infty_t(L^\infty)$, we conclude that $X(t) \leq C_t$ and by this way we have proved that:

$$\|\rho\|_{L^\infty_T(B^\epsilon_{\infty,\infty})} \leq C_{0,T}, \tag{6.86}$$

where $C_{t,0}$ depnds only of the time T and the initial data.

6.0.2 Control of $\rho \in L^1(B^1_{N,1})$

In this case, we need to show for the sequel that $\rho \in L^\infty(B^1_{N,1})$, and for this we proceed exactly as previous. Indeed as $\rho_0 \in B^1_{N,1}$, by proceding as in the previous section we can show that $\rho \in L^\infty_T(B^1_{N,1})$. Next by using proposition 2.6, we obtain the fact that:

$$\|\rho\|_{L^\infty_T(B^1_{N,1})} \leq e^{CV(T)}(1 + \|q_0\|_{B^1_{N,1}}), \tag{6.87}$$

with $V(T) = \int_0^T (\|\nabla v_1(\tau)\|_{B^{1+\epsilon}_{\infty,\infty}} + \|q(\tau)\|_{B^\epsilon_{\infty,\infty}})d\tau$.

6.1 Uniqueness

We now discuss the uniqueness of the solutions of theorem 1.2.For this we want use the result of P. Germain [31] which is a result of weak-strong uniqueness. In the sequel we will note (ρ_1, u_1) the solution of the theorem 1.2 which exits on the time interval $[0, T_0]$. We have shown that our solution check $\rho \in L^\infty(L^\infty)$. By theorem 1.2, we obtain that our solution verify the following inequalities:

$$\begin{aligned}
&\sup_{0<t\leq+\infty} \int_{\mathbb{T}^N} [\frac{1}{2}\rho(t,x)|u(t,x)|^2 + |P(\rho(t,x))| + \sigma(t)|\nabla u(t,x)|^2 dx \\
&+ \sup_{0<t\leq+\infty} \int_{\mathbb{T}^N} [\frac{1}{2}\rho(t,x) f(t)^N (\rho|\dot{u}(t,x)|^2 + |\nabla\omega(t,x)|^2)dx \\
&+ \int_0^{+\infty}\int_{\mathbb{T}^N} [|\nabla u|^2 + f(s)\rho|\dot{u}|^2 + |\omega|^2) + \sigma^N |\nabla \dot{u}|^2]dxdt \\
&\qquad\qquad\qquad\qquad \leq C(C_0 + C_f)^\theta,
\end{aligned} \tag{6.88}$$

and we obtain moreover:

$$\begin{cases}
\sqrt{\rho}\partial_t u \in L^2_t(L^2(\mathbb{T}^N)), \\
\sqrt{t}\mathcal{P}u \in L^2_T(H^2(\mathbb{T}^N)), \\
\sqrt{t}G = \sqrt{t}[(\lambda + 2\mu)\mathrm{div}u - P(\rho)] \in L^2_T(H^1(\mathbb{T}^N)), \\
\sqrt{t}\nabla u \in L^\infty_T(L^2(\mathbb{T}^N)),
\end{cases} \tag{6.89}$$

Now by the result of P. Germain in [31], we are able to prove that $(\rho, u) = (\rho_1, u_1)$ on $[0, T_0]$. To see this we have just to verify that $(\rho_1, u_1$ verify the conditions of the theorem 2.2 of [31]. For simplicity we prove only the result for $N = 3$. We know that $\nabla\rho_1 \in L^\infty(B^\epsilon_{N,1}) \hookrightarrow L^\infty_{T_0}(L^N)$ and $\nabla u_1 \in L^1_{T_0}(L^\infty)$. The main thing is to see that $\sqrt{t}\dot{u}_1 \in L^2_{T_0}(L^N)$.
We recall by Gagliardo-Nirenberg inequalities that:

$$\sqrt{t}\|\dot{u}_1\|_{L^3} \leq (t^{\frac{1}{4}-\frac{\epsilon}{2}}\|\dot{u}_1\|_{L^2})^{\frac{1}{2}}(t^{\frac{3}{2}-\frac{\epsilon}{2}}\|\nabla\dot{u}_1\|_{L^2})^{\frac{1}{2}} t^{\frac{\epsilon}{2}}.$$

From the inequalities (1.13), we deduce that $\sqrt{t}\dot{u}_1 \in L^2(L^3)$.

7 Proof of theorem 1.3

7.1 How to obtain a regularizing effect on v_1 when $\rho \in L^\infty(L^q)$

For the moment, we don't give conditions on q, we will precise his value in the sequel of the proof.
We want now use our change of variable v_1 introduced in the previous sections. The interest of this new variable is to "kill" in a certain way the coupling velocity-pressure. In this goal, we can now rewrite the momentum equation of system (1.1). We obtain then the following equation where we have set $\nu = 2\mu + \lambda$:

$$\rho\partial_t u + \rho u\cdot\nabla u - \mu\Delta\big(u - \frac{1}{\nu}v\big) - (\lambda+\mu)\nabla\mathrm{div}\big(u - \frac{1}{\nu}v\big) = \rho g,$$

where we recall that $v = (\Delta)^{-1}(\nabla P(\rho))$ with $(\Delta)^{-1}$ the inverse Laplacian with zero mean value on $\mathbb{T}^N$. As $v_1 = u - \frac{1}{\nu}v$ we have:

$$\rho\partial_t v_1 + \rho u\cdot\nabla u - \mu\Delta v_1 - (\lambda+\mu)\nabla\mathrm{div} v_1 = \rho g - \frac{1}{\nu}\rho\partial_t v.$$

As $\mathrm{div} v = P(\rho) - \int_{\mathbb{T}^N} P(\rho)dx$, from the transport equation we obtain:

$$\begin{aligned}\mathrm{div}\partial_t v &= -P^{'}(\rho)\rho\mathrm{div}u - \nabla P(\rho)\cdot u + \widetilde{P'(\rho)\rho\mathrm{div}u} + \widetilde{\nabla P(\rho)\cdot u}\\ &= -\mathrm{div}(P(\rho)u) + (P(\rho) - \rho P^{'}(\rho))\mathrm{div}u - \widetilde{P(\rho)\mathrm{div}u} + \widetilde{\rho P'(\rho))\mathrm{div}u}.\end{aligned}$$

In the sequel we will need to use the Bogovskii operator that we note Λ^{-1} (see [56] p168 for a definition), we obtain then:

$$\partial_t v = \Lambda^{-1}\big(-\mathrm{div}(P(\rho)u) + (P(\rho) - \rho P^{'}(\rho))\mathrm{div}u - \widetilde{P(\rho)\mathrm{div}u} + \widetilde{\rho P'(\rho))\mathrm{div}u}\big). \quad (7.90)$$

We get finally

$$\rho\partial_t v_1 - \mu\Delta v_1 - (\lambda+\mu)\nabla\mathrm{div} v_1 = \rho g - \rho u\cdot\nabla u - \frac{1}{\nu}\rho\partial_t v. \quad (7.91)$$

We set $f(t) = \min(t,1)$ and we remark that $f(0) = 0$. We multiply then (7.91) by $f(t)\partial_t v_1$ and integrating on $(0,T)\times\mathbb{T}^N$ we obtain where $\xi = \mu + \lambda$:

$$\begin{aligned}&\int_0^t\int_{\mathbb{T}^N} f(s)\rho|\partial_s v_1|^2 dxds + \frac{1}{2}\int_{\mathbb{T}^N} f(t)\big(\mu|\nabla v_1(t,x)|^2 + \xi(\mathrm{div}v_1)^2(t,x)\big)dx \le\\ &\qquad\int_0^t\int_{\mathbb{T}^N} f^{'}(s)(\mu|\nabla v_1(t,x)|^2 + \xi(\mathrm{div}v_1)^2(t,x)\big)dxds \quad (7.92)\\ &\qquad+\int_0^t\int_{\mathbb{T}^N}\rho u\cdot\nabla u\, f(s)\partial_t v_1 dxds + \int_0^t\int_{\mathbb{T}^N}(\rho g - \frac{1}{\nu}\rho\partial_s v)f(s)\partial_s v_1 dxds.\end{aligned}$$

We have then as $P(\rho)\in L^2_t(L^2)$ and by bootstrap:

$$\begin{aligned}&\int_0^t\int_{\mathbb{T}^N} f(s)\rho|\partial_s v_1|^2 dxds + \frac{1}{2}\int_{\mathbb{T}^N} f(t)\big(\mu|\nabla v_1(t,x)|^2 + \xi(\mathrm{div}v_1)^2(t,x)\big)dx \le\\ &\qquad C\big(1 + \int_0^t\int_{\mathbb{T}^N} f(s)\rho|u\cdot\nabla v_1|^2 dxds + \int_0^t\int_{\mathbb{T}^N} f(s)\rho|u\cdot\nabla v|^2 dxds \quad (7.93)\\ &\qquad+\int_0^t\int_{\mathbb{T}^N} f(s)\rho(|g|^2 + |\partial_s v|^2)dxds\big).\end{aligned}$$

We have next to control the terms on the right hand side of (7.92). In the sequel we will for simplicity treat only the case $N=3$. We can recall now that from the works of A. Mellet and A. Vasseur in [53], we control the velocity u in $L^\infty(L^\infty)$ as we have assume that $\rho\in L^\infty(L^{3\gamma+\epsilon})$ with $\epsilon>0$. In fact from the inequality (4.42), we could get a gain on $\rho^{\frac{1}{p}}u\in L^\infty(L^p)$ with p arbitrary big if $P(\rho)\in L^p(L^{\frac{3p}{p+1}})$ and in our case il will be the case as we assume that $P(\rho)\in L^\infty(L^3)$. For the simplicity of the calculus we assume that $u\in L^\infty_t(L^\infty)$. In fact we have only $\rho^{\frac{1}{p}}u\in L^\infty(L^p)$ for p arbitrarily large, but in the sequel all the expressions to treat are of the form ρu. It would suffice to apply the Hölder's inequalities with $\rho^{1-\frac{1}{p}}(\rho^{\frac{1}{p}}u)$. We begin now with:

$$\int_0^t\int_{\mathbb{T}^N}u\cdot\nabla u\,\rho f(s)\partial_s v_1dxds=\int_0^t\int_{\mathbb{T}^N}\big(u\cdot\nabla v_1+u\cdot\nabla(\Delta)^{-1}\nabla(P(\rho))\big)\,\rho f(s)\partial_s v_1dxds.$$

Estimates on the term $\int_0^t\int_{\mathbb{T}^N}u\cdot\nabla(\Delta)^{-1}\nabla(P(\rho))\,\rho f(s)\partial_t v_1dxds$

We start with treating the easier term:

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{T}^N}u\cdot\nabla(\Delta)^{-1}\nabla(P(\rho))\,\rho f(s)\partial_t v_1dxds\Big|\\&\leq C_\alpha\|\sqrt{f(t)\rho}\,uP(\rho)\|^2_{L^2_t(L^2(\mathbb{T}^N))}+\alpha\|\sqrt{f(t)\rho}\partial_t v_1\|^2_{L^2_t(L^2(\mathbb{T}^N))},\\&\leq C_{\alpha,t}\|\rho\|^{2\gamma}_{L^\infty(L^q)}\|u\|^2_{L^\infty(L^\infty)}\|\rho\|_{L^\infty(L^q)}+\alpha\|\sqrt{f(t)\rho}\partial_t v_1\|^2_{L^2_t(L^2(\mathbb{T}^N))},\end{aligned}\tag{7.94}$$

with $\frac{\gamma}{q}+\frac{1}{2q}\leq\frac{1}{2}$ (let $q\geq 2\gamma+1$).

Regularizing effect on Δv_1

We want here using the regularizing effect on v_1. To do it we use the momentum equation (7.91) and we have:

$$\mu\Delta v_1+(\lambda+\mu)\nabla{\rm div}v_1=\rho\partial_t v_1+\rho u\cdot\nabla v_1+\rho u\cdot\nabla(\Delta)^{-1}\nabla(P(\rho))-\rho g+\frac{\rho}{\nu}\partial_t v.\tag{7.95}$$

We want use now the ellipticity of (7.95), for this we recall that we can hope only $\sqrt{\rho f(t)}\partial_t v_1\in L^2(L^2)$. We set then $\frac{1}{p}=\frac{1}{2}+\frac{1}{2q}$ and we have:

$$\begin{aligned}\|\Delta v_1\|_{L^p}\leq\|\rho\|^{\frac{1}{2}}_{L^q}\|\sqrt{\rho}\partial_t v_1\|_{L^2}+\|u\|_{L^\infty}\|\rho\|_{L^q}\|\nabla v_1\|_{L^{q_1}}+\|\rho\|_{L^q}\|u\|_{L^\infty}\|\rho\|^\gamma_{L^q}\\+\|\rho\|_{L^q}(\|\partial_t v\|_{L^{q_1}}+\|g\|_{L^{q_1}}),\end{aligned}\tag{7.96}$$

with the following conditions: $\frac{1}{q}+\frac{1}{q_1}\leq\frac{1}{p}$ (let $\frac{1}{q_1}=\frac{1}{2}-\frac{1}{2q}$) and $\frac{1}{q}+\frac{\gamma}{q}\leq\frac{1}{p}$ (let $q\geq 2\gamma+1$). Next by Gagliardo-Nirenberg, we have

$$\|\nabla v_1\|_{L^{q_2}}\leq\|\nabla v_1\|^\epsilon_{L^2}\|\Delta v_1\|^{1-\epsilon}_{L^p},\tag{7.97}$$

with: $\frac{1}{q_2}=\frac{1}{2}+\frac{1}{2q}-\frac{1}{3}+\frac{\epsilon}{3}$ where $\epsilon\in]0,1[$. We can now estimate the following term $\int_0^t\|\sqrt{f(s)\rho}u\cdot\nabla v_1\|^2_{L^2}ds$.

Estimate on the term $\int_0^t \|\sqrt{f(s)\rho}u\cdot\nabla v_1\|_{L^2}^2 ds$

We have then by using (7.97) where ϵ is defined such that $\frac{1}{q_1}=\frac{1}{2}+\frac{1}{2q}-\frac{1}{3}+\frac{\epsilon}{3}$ and (7.96):

$$
\begin{aligned}
\|\sqrt{f(s)\rho}u\cdot\nabla v_1\|_{L^2}^2 &\leq \|\rho\|_{L^q}^2\|u\|_{L^\infty}^2\|\sqrt{f(s)}\nabla v_1\|_{L^{q_1}}^2,\\
&\leq \|\rho\|_{L^q}^2\|u\|_{L^\infty}^2 f(s)\|\nabla v_1\|_{L^2}^{2\epsilon}\|\Delta v_1\|_{L^p}^{2(1-\epsilon)},\\
\leq C\|\rho\|_{L^q}^2\|u\|_{L^\infty}^2 f(s)\|\nabla v_1\|_{L^2}^{2\epsilon}&\big(\|\rho\|_{L^q}^{\frac{1}{2}}\|\sqrt{\rho}\partial_t v_1\|_{L^2}+\|\rho u\cdot\nabla v_1\|_{L^2}\\
&+\|\rho\|_{L^q}\|u\|_{L^\infty}\|\rho\|_{L^q}^{\gamma}+\|\rho\|_{L^q}(\|\partial_t v\|_{L^{q_1}}+\|g\|_{L^{q_1}})\big)^{2(1-\epsilon)},
\end{aligned}
$$

Next by Young inequality, we get:

$$
\begin{aligned}
\|\sqrt{f(s)\rho}u\cdot\nabla v_1\|_{L^2}^2 &\leq C_\alpha f(s)\|\rho\|_{L^q}^{\frac{2}{\epsilon}}\|u\|_{L^\infty}^{\frac{2}{\epsilon}}\|\nabla v_1\|_{L^2}^2(1+\|\rho\|_{L^q}^{\frac{1-\epsilon}{\epsilon}})\\
&+\alpha\|\sqrt{f(s)\rho}\partial_t v_1\|_{L^2}^2+\alpha\|\sqrt{f(s)}\rho u\cdot\nabla v_1\|_{L^2}^2+\alpha\|\rho\|_{L^q}^2\|u\|_{L^\infty}^2\|\rho\|_{L^q}^{2\gamma}\\
&+\alpha\|\rho\|_{L^q}^2(\|\sqrt{f(s)}\partial_t v\|_{L^{q_1}}^2+\|g\|_{L^{q_1}}^2),
\end{aligned}
\tag{7.98}
$$

Here α is very small and C_α can be very big. Now from (7.90), we have:

$$
\|\sqrt{f(s)}\partial_t v_1\|_{L^{q_1}}^2 \leq \|\sqrt{f(s)}P(\rho)u\|_{L^{q_1}}^2+\|\sqrt{f(s)}\Lambda^{-1}\big(P(\rho)-\rho P'(\rho)\big)\mathrm{div}u\big)\|_{L^{q_1}}^2.
$$

We have then:

$$
\|\sqrt{f(s)}\Lambda^{-1}\big(P(\rho)-\rho P'(\rho)\big)\mathrm{div}u\big)\|_{L^{q_1}}^2 \leq f(s)\|\rho\|_{L^q}^{2\gamma}\|\mathrm{div}u\|_{L^2}^2, \tag{7.99}
$$

where $\frac{\gamma}{q}+\frac{1}{2}-\frac{1}{3}\leq\frac{1}{q_1}$ (let $q\geq 3\gamma+\frac{3}{2}$). We treat similarly the term $\|\sqrt{f(s)}P(\rho)u\|_{L^{q_1}}^2$. Now from (7.93), (7.94), (7.98) and (7.99), we have:

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{T}^N} f(s)\rho|\partial_s v_1|^2 dxds+\frac{1}{2}\int_{\mathbb{T}^N} f(t)\big(\mu|\nabla v_1(t,x)|^2+\xi(\mathrm{div}v_1)^2(t,x)\big)dx\leq\\
&\quad C\big(1+\int_0^t\int_{\mathbb{T}^N} f(s)\rho|u\cdot\nabla v_1|^2 dxds+\int_0^t\int_{\mathbb{T}^N} f(s)\rho(|g|^2+|\partial_s v|^2)dxds\big).
\end{aligned}
\tag{7.100}
$$

By proceeding as in the previous terms for $\int_0^t\int_{\mathbb{T}^N} f(s)\rho(|g|^2+|\partial_s v|^2)dxds$, we conclude that if $q\geq 3\gamma+\frac{3}{2}$ then:

$$
\int_0^t\int_{\mathbb{T}^N} f(s)\rho|\partial_s v_1|^2 dxds+\frac{1}{2}\int_{\mathbb{T}^N} f(t)\big(\mu|\nabla v_1(t,x)|^2+\xi(\mathrm{div}v_1)^2(t,x)\big)dx\leq C_t,
\tag{7.101}
$$

where C_t depends on t.

Control on the norm $\rho\in L^\infty$

Next we come back to equation (3.35) to get a control in norm L^∞ on the density, more precisely we have:

$$
\begin{aligned}
\log(\rho(t,x)) &\leq \log(\|\rho_0\|_{L^\infty})+C\|(\Delta)^{-1}\mathrm{div}m_0\|_{L^\infty}+C\|(\Delta)^{-1}\mathrm{div}(\rho u)\|_{L^\infty}\\
&+C\int_0^t\frac{1}{\sqrt{f(s)}}\|[\sqrt{f(s)}u_j,R_iR_j](\rho u_i)(s,\cdot)\|_{L^\infty}ds.
\end{aligned}
\tag{7.102}
$$

Easily, we have by Sobolev embedding and (4.42) with $\epsilon > 0$:

$$\begin{aligned}\|(\Delta)^{-1}\text{div}(\rho u)\|_{L^\infty} &\leq \|\rho u\|_{L^\infty(L^{3+\epsilon})}\\ &\leq \|\rho\|_{L^\infty(L^q)}\|u\|_{L^\infty(L^\infty)},\end{aligned}$$

as $q > 3$.
We recall then from the previous section that $\sqrt{f(s)}\Delta v_1 \in L^2(L^p)$ with $\frac{1}{p} = \frac{1}{2} + \frac{1}{2q}$, by Gagliardo-Nirenberg inequality we have:

$$\|\nabla v_1\|_{L^{q_2}} \leq \|\Delta v_1\|_{L^p}^{1-\epsilon}\|\nabla v_1\|_{L^2}^{\epsilon},$$

with $\frac{1}{q_2} = \frac{1}{2} + \frac{1}{2q} - \frac{1}{3} + \frac{\epsilon}{3}$. We have then by using the results of R. Coifman et al in [19]:

$$\begin{aligned}(f(s))^{\frac{1}{2}-\frac{\epsilon}{2}}\|[(v_1)_j, R_iR_j](\rho u_i)(s,\cdot)\|_{W^{1,\alpha}} \leq \|\nabla v_1\|_{L^2}^{\epsilon}\|\sqrt{f(s)}\Delta v_1\|_{L^p}^{1-\epsilon}\\ \times \|\rho\|_{L^q}\|u\|_{L^\infty},,\end{aligned} \tag{7.103}$$

with $\frac{1}{q} + \frac{1}{q_2} = \frac{1}{2} + \frac{1}{2q} - \frac{1}{3} + \frac{\epsilon}{3} < \frac{1}{3}$, which means for ϵ enough small $q > 9$. We have finally:

$$\begin{aligned}|\int_0^t \frac{1}{f(s)^{\frac{1}{2}-\frac{\epsilon}{2}}} f(s)^{\frac{1}{2}-\frac{\epsilon}{2}}\|[(v_1)_j, R_iR_j](\rho u_i)(s,\cdot)\|_{L^\infty}ds| \leq C\|\rho\|_{L^\infty(L^q)}\|u\|_{L^\infty(L^\infty)}\\ \|\nabla v_1\|_{L^2(L^2)}^{\epsilon}\|\sqrt{f(s)}\Delta v_1\|_{L^2(L^p)}^{1-\epsilon}.\end{aligned}$$

We proceed similarly for the term $\|[(v_j, R_iR_j](\rho u_i)(s,\cdot)\|_{L^1_s(L^\infty)}$. This term is crucial because it gives the value of q that we must choose. Indeed we have the results of R. Coifman et al in [19] if $q < 3\gamma$:

$$\|[(\Delta)^{-1}\nabla(P(\rho))_j, R_iR_j](\rho u_i)(s,\cdot)\|_{W^{1,\beta}} \leq \|\rho\|_{L^q}^{\gamma}\|\|u\|_{L^\infty}\|\rho\|_{L^q},$$

with $\frac{1}{q} + \frac{\gamma}{q} < \frac{1}{3}$. We need then that $q > 3(\gamma+1)$ and that $\rho \in L_t^{\gamma+1}(L^q)$.
If we summarize all the inequalities on q, we need that $\rho \in L^\infty(L^q)$ with $q \geq 3\gamma + \frac{3}{2}$ this from the previous section. Moreover as we want that $\rho^{\frac{1}{p}}u \in L^\infty(L^p)$ for any arbitrarly large p, we need that $\rho \in L^\infty(L^{3\gamma})$. From this section we have shown that we must have $\rho \in L^\infty(L^{9+\epsilon}) \cap L^{\gamma+1}(L^{3(\gamma+1+\epsilon)})$.
Finally under theses conditions we control $|\log\rho|1_{\{\rho\geq 1\}} \in L^\infty$ by the fact that $q > 3(\gamma+1)$ and $q > 9$. We have then obtained the fact that $\rho \in L^\infty$. Now it is easy to control $\frac{1}{\rho} \in L^\infty(L^\infty)$, it suffices to minore $\log\rho$. To do this we have just to proceed similarly than previous and to observe taht $P(\rho) \in L^\infty$.
It means that $\rho \in L^\infty$ and $\frac{1}{\rho} \in L^\infty$, from theorem 1.2 we have seen that the inequality (1.10) and (1.11) are preserved during the time if $\rho \in L^\infty$. We can now assume that (ρ, u) verify (1.10) and (1.11). We want now prove the uniqueness of this solution.

7.2 Uniqueness

We want prove now the result of uniqueness. To do it we want use the result of P. Germain in [31]. In the sequel we will note (ρ_1, u_1) the solution of the theorem 1.2 which exits on the time interval $[0,T_0]$. We have shown that our solution check $\rho\in L^\infty(L^\infty)$. By theorem 1.2, we obtain that our solution verify the following inequalities:

$$\begin{aligned}&\sup_{0<t\leq+\infty}\int_{\mathbb{T}^N}[\frac{1}{2}\rho(t,x)|u(t,x)|^2+|P(\rho(t,x))|+f(t)|\nabla u(t,x)|^2dx\\&+\sup_{0<t\leq+\infty}\int_{\mathbb{T}^N}[\frac{1}{2}\rho(t,x)f(t)^N(\rho|\dot{u}(t,x)|^2+|\nabla\omega(t,x)|^2)dx\\&+\int_0^{+\infty}\int_{\mathbb{T}^N}[|\nabla u|^2+f(s)(\rho|\dot{u}|^2+|\nabla\omega|^2)+f^N(t)|\nabla\dot{u}|^2]dxdt\\&\hspace{6cm}\leq C(C_0+C_g)^\theta,\end{aligned}\tag{7.104}$$

and:

$$\begin{aligned}&\sup_{t>0}\int_{\mathbb{T}^N}[\frac{1}{2}\rho(t,x)|u(t,x)|^2+|P(\rho(t,x))|+t^{2-\frac{N}{2}-\epsilon}|\nabla u(t,x)|^2dx\\&+\sup_{0<t\leq T_0}\int_{\mathbb{T}^N}[\frac{1}{2}\rho(t,x)t^\sigma(\rho|\dot{u}(t,x)|^2+|\nabla\omega(t,x)|^2)dx\\&+\int_0^{+\infty}\int_{\mathbb{T}^N}[|\nabla u|^2+t^{2-\frac{N}{2}-\epsilon}|\dot{u}|^2+t^\sigma|\nabla\dot{u}|^2]dxdt\leq C(C_0+C_f)^\theta,\end{aligned}\tag{7.105}$$

where:

$$\begin{cases}\sigma=3-\dfrac{N}{2}-\epsilon,\quad\text{if }\ N=2,\\ \sigma=\max(3-\dfrac{N}{2}-\epsilon,6-\dfrac{3}{2}N-3\epsilon),\quad\text{if }\ N=3,\end{cases}\tag{7.106}$$

and:

$$\sup_{0\leq t\leq T_0}\|u(t,\cdot)\|_{H^{\frac{N}{2}-1+\epsilon}}\leq CC_0^\theta\ \text{ and }\ \sup_{0\leq t\leq T_0}\|\rho^{\frac{1}{r}}u(t,\cdot)\|_{L^r}\leq C(r)C_0^\theta,\tag{7.107}$$

with $r\in(1,4]$. Moreover we have:

$$\begin{cases}\sqrt{\rho}\partial_t u\in L^2_t(L^2(\mathbb{T}^N)),\\ \sqrt{t}\mathcal{P}u\in L^2_T(H^2(\mathbb{T}^N)),\\ \sqrt{t}G=\sqrt{t}[(\lambda+2\mu)\text{div}u-P(\rho)]\in L^2_T(H^1(\mathbb{T}^N)),\\ \sqrt{t}\nabla u\in L^\infty_T(L^2(\mathbb{T}^N)),\end{cases}\tag{7.108}$$

From the previous section, we have seen that $\rho\in L^\infty$ and $\frac{1}{\rho}\in L^\infty$. To finish, by proceeding as in the section **??**, we are able to show that $\rho\in L^\infty(B^{1+\epsilon}_{N,1})$ with $\epsilon>0$.
Now by the result of P. Germain in [31], we are able to prove that if (ρ,u) and (ρ_1,u_1) are two such solutions verifying all the previous estimates on $[0,T_0]$ then $(\rho,u)=(\rho_1,u_1)$ on $[0,T_0]$. To see this we have just to verify that (ρ_1,u_1) verify the conditions of the theorem 2.2 of [31]. For simplicity we prove only the result for $N=3$. We know that $\nabla\rho_1\in L^\infty(B^\epsilon_{N,1})\hookrightarrow L^\infty_{T_0}(L^N)$ and $\nabla u_1\in L^1_{T_0}(L^\infty)$. In fact to prove that $\nabla u_1\in L^1_{T_0}(L^\infty)$, we have just to observe that $\nabla u_1=\nabla(v_1)_1+\nabla v_1$, but by following the same process as in the previous section, we know that $\nabla(v_1)_1\in L^1_{T_0}(L^\infty)$. Nex we recall that $\nabla v_1=\nabla(\Delta)^{-1}\nabla P(\rho_1)$ and as $\rho_1\in L^\infty_{T_0}\cap L^\infty_{T_0}(B^{1+\epsilon}_{N,1})$ we have $P(\rho_1)\in L^\infty_{T_0}(B^{1+\epsilon}_{N,1})$, and so

$v_1 \in L^\infty_{T_0}(B^{1+\epsilon}_{N,1})$ which means that $\nabla v_1 \in L^\infty_{T_0}(B^\epsilon_{N,1}) \hookrightarrow L^\infty_{T_0}(L^\infty)$. We conclude then that $\nabla u_1 \in L^\infty_{T_0}(L^\infty)$.
The second thing is that we control $\nabla\rho_1 \in L^{\inf}(L^3)$ as in the theorem 2.2 of [31]. The main thing is now to see that $\sqrt{t}\dot{u}_1 \in L^2_{T_0}(L^3)$.
We recall by Gagliardo-Nirenberg inequalities that:

$$\sqrt{t}\|\dot{u}_1\|_{L^3} \leq (t^{\frac{1}{4}-\frac{\epsilon}{2}}\|\dot{u}_1\|_{L^2})^{\frac{1}{2}}(t^{\frac{3}{4}-\frac{\epsilon}{2}}\|\nabla\dot{u}_1\|_{L^2})^{\frac{1}{2}}t^{\frac{\epsilon}{2}}.$$

From the inequalities (7.105), we deduce that $(t^{\frac{1}{4}-\frac{\epsilon}{2}}\|\dot{u}_1\|_{L^2})^{\frac{1}{2}} \in L^4_{t_0}(L^4)$ and $(t^{\frac{3}{4}-\frac{\epsilon}{2}}\|\nabla\dot{u}_1\|_{L^2})^{\frac{1}{2}} \in L^4_{T_0}(L^4)$ which means that $\sqrt{t}\dot{u}_1 \in L^2_{T_0}(L^3)$.
We have proved then that by using the results of [31] that $(\rho, u) = (\rho_1, u_1)$ on $[0, T_0]$ for all $T_0 > 0$ which conclude the proof.

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