Analysis of Generalized Eigenvalue Problems

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A **standard eigenvalue problem** is of the form

\[ Au = \mu u \quad \text{in} \ L. \quad \text{(EP)} \]

A **generalized eigenvalue problem** is of the form

\[ Au = \mu Bu \quad \text{in} \ L. \quad \text{(GEP)} \]

If \( A \) is invertible then the generalized eigenvalue problem is of the standard form

\[ \lambda u = \left( A^{-1} B \right) u \quad \text{in} \ L, \]

for \( \lambda \neq 0 \).
Aims

1. Description of the spectra of a family of GEPs.
2. Asymptotics of the spectra of a family of GEPs.

Illustration of the ideas using a toy problem:

Simple model for the motion of fuel rods in a core of a nuclear power plant.
1. Fresh-Up in Spectral Theory and Functional Analysis: ➤ Eberhard
   ➤ Fundamental Structures.
   ➤ Parts of the Spectrum and their description.
   ➤ Characterization of selfadjoint operators.

2. Motion of Tubes in Fluid: ➤ Rike
   ➤ Physical Modeling of the problem.
   ➤ Mathematical Formulation.
   ➤ Bloch Wave Method.
   ➤ Decomposition into independent rods.

3. GEPs and their Asymptotics: ➤ Eberhard
   ➤ Mathematical Structure of the Problem.
   ➤ Properties and Dependence: Finite number of rods.
   ➤ Asymptotics of the Problem: Infinite number of rods.
Fundamental Structures:
- Spectrum $\sigma$, Resolvent set $\varrho$, . . . .
- (Quadratic) Forms and Associated Operators.

Parts of the Spectrum and their Description:
- Discrete Spectrum $\sigma_d$.
- Minimax Principle.
- Monotonicity of the Eigenvalues.

Characterization of selfadjoint operators:
- Functional Calculus.
- Spectral Resolution.
1 Fundamental structures

1.1 Spectrum $\sigma$, Resolvent set $\varrho$, …

$A : H \supseteq D(A) \longrightarrow H$ a linear operator. Then consider

$$(\lambda - A) u = f \quad \text{in } H.$$  

Three questions:

Existence: For every $f \in H$ exists a $u \in H$. $\Rightarrow$ Surjectivity of $A_\lambda := \lambda - A$.

Uniqueness: For every $f \in H$ exists at most one $u \in H$. $\Rightarrow$ Injectivity of $A_\lambda$.

Stability: $u$ depends continuously on $f$. 

Asymptotics of Non-local Spectral Problems (6/65)
Hadamard’s Questions:

Existence: For every \( f \in H \) exists a \( u \in H \) \( \Rightarrow \) Surjectivity of \( A_\lambda \).

Uniqueness: For every \( f \in H \) exists at most one \( u \in H \) \( \Rightarrow \) Injectivity of \( A_\lambda \).

Stability: \( u \) depends continuously on \( f \).

Closed Operators \( \Rightarrow \) Stability for free.
Definition 1.1. The Resolvent Set $\varrho(A)$ is

$$\varrho(A) = \left\{ \lambda \in \mathbb{C} \left| A_\lambda \text{ is bijective and } R_\lambda := (A_\lambda)^{-1} \text{ is continuous} \right. \right\}.$$ 

The Spectrum $\sigma(A)$ is given by $\sigma(A) := \mathbb{C} \setminus \varrho(A)$.

If $A_\lambda$ isn’t injective then $\lambda$ is an **Eigenvalue**. The **Point Spectrum** $\sigma_p(A)$ is the set of eigenvalues.

Remark 1.2. Always $\sigma_p(A) \subseteq \sigma(A)$, but in most cases $\sigma(A) \neq \sigma_p(A)$.

- If $\text{dim } H < \infty$ then $\sigma(A) = \sigma_p(A)$.
- If $A$ is finite or compact then $\sigma(A) = \sigma_p(A) \cup \{0\}$. 

Asymptotics of Non-local Spectral Problems (7/65)
1.2 Forms and associated Operators

Definition 1.3 (Quadratic Form). \( H \) a Hilbert-space. \( \mathcal{A} \in \text{Lin}(H, H^*) \) is called a Sesquilinear Form, a.k.a Form. The mapping

\[
H \ni u \mapsto \mathcal{A}[u] := \mathcal{A}[u, u] := \mathcal{A}u(u)
\]

is called the Quadratic Form.

Notation 1.4. \( \mathcal{A} \) Form and Quadratic Form determine each other uniquely by (1.3) and the so called Polarization Principle

\[
\mathcal{A}[u, v] = \frac{1}{4} \sum_{k=0}^{3} i^k \mathcal{A}[u + i^k v].
\]
Definition 1.5 (Associated Operator). Let $\mathcal{A}$ be a form on $H$, $L$ be an other H-space s.t. $H$ is dense in $L$. The inner product on $L$ is denoted by $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_L$. Then the associated operator $A$ (on $L$) is given by

$$\mathcal{A} u = A u \quad \text{for } u \in D(A),$$

where $D(A) = \left\{ u \in L \mid u \in H \text{ and } \mathcal{A} u \in L \right\}$. The space $H$, considered as subset of $L$, is denoted by $D(A)$.

Remark 1.6. Actually $\mathcal{A} u = A u$ is short for $\mathcal{A} \iota_{H \subseteq L} u = \iota_{L \subseteq H^*} A u$. $A$ is the maximal operator that makes the following diagram commutative.

Similarly $u \in H$ is short for $u \in \iota_{H \subseteq L} H$ and $\mathcal{A} u \in L$ stands for $\mathcal{A} u \in \iota_{L \subseteq H^*} L$. For all practical purposes, each $\iota$ is basically the identity.
Example 1.7 (Laplacian). Let $L = L^2(\Omega)$ and $A$ be formally given by Dirichlet’s Integral

$$A[u] := \int_{\Omega} |Du|^2.$$ 

Let $H := H^1(\Omega)$. The associated operator is called the Neumann-Laplacian on $\Omega$, denoted by $-\Delta^N_\Omega$.

If $H := H^1_0(\Omega)$ the associated operator is called Dirichlet-Laplacian on $\Omega$, referred to as $-\Delta^D_\Omega$.

If $H := H^1_{\text{per}}(\Omega)$ the associated operator is called Laplacian on $\Omega$ with periodic boundary conditions, denoted by $-\Delta^\text{per}_\Omega$.

If $L$ and $H$ are known, every Laplacian is referred to as $-\Delta$. 
For $u \in H$ the Rayleigh Quotient (of $A$ in $L$) is given by

$$
R[u] := \frac{A[u, u]}{\langle u, u \rangle}.
$$

The set of values of the Rayleigh Quotient is called the Numerical Range of $A$. It is denoted by

$$
W(A) := \left\{ R[u] \bigg| 0 \neq u \in H \right\}.
$$

If $W(A) \subseteq \left\{ \lambda \in \mathbb{C} \bigg| c \leq \text{Re} \lambda \right\}$ for some $c \in \mathbb{R}$ then $A$ is called semibounded (from below). Short $c \leq \text{Re} A$. $A$ is called symmetric or real valued if $A[u] \in \mathbb{R}$ for $u \in H$. In this case semiboundedness is denoted by $c \leq A$. 

Asymptotics of Non-local Spectral Problems (11/65)
Theorem 1.8 (Lax-Milgram). If $0 < c \leq \text{Re} \mathcal{A}$ then $\mathcal{A}$ is bijective.

Example 1.9 (Laplacians). Let $\mathcal{A}$ be given by $\mathcal{A}[u] := \int_{\Omega} |Du|^2$.

Let $H := H^1(\Omega)$. By construction $\mathcal{A}[u] \geq 0$. Since $\mathcal{A}$ vanishes on constants $C \leq H$, $\mathcal{A}$ is not strictly positive. But

$$0 < c \leq \mathcal{A} \text{ in } H/C.$$

Let $H := H^1_0(\Omega)$. Since $C \not\leq H$, the form $\mathcal{A}$ is strictly positive. Hence,

$$0 < c \leq \mathcal{A} \text{ in } H.$$

Let $H := H^1_{\text{per}}(\Omega)$. Since $C \leq H$, $\mathcal{A}$ is not strictly positive. But

$$0 < c \leq \mathcal{A} \text{ in } H/C.$$
Hausdorff called $W(\mathcal{A})$ *Wertevorrat* of $\mathcal{A}$
or *Wertebereich* of $\mathcal{A}$ $\Rightarrow$ Halmos, Werner,
2 Description of the Spectrum

2.1 Discrete Spectrum

Definition 2.1. Let $A$ be a selfadjoint operator. The discrete spectrum is denoted by

$$
\sigma_d(A) = \left\{ \lambda \in \sigma_p(A) \mid \dim \text{Eig}_\lambda(A) < \infty \text{ and } \text{dist}(\lambda, \sigma(A) \setminus \{\lambda\}) > 0 \right\}.
$$

The essential spectrum is given by

$$
\sigma_e(A) = \sigma(A) \setminus \sigma_d(A).
$$
2.2 Minimax Principle

Remark 2.2 (Rayleigh Principle). Let $A$ be a selfadjoint operator on $\mathbb{C}^n$. Then the smallest and largest eigenvalues are given by

$$
\lambda_1 = \min W(A) = \min_{0 \neq \phi \in \mathbb{C}^n} \frac{A\phi \cdot \phi}{\phi \cdot \phi} \quad \text{and} \\
\lambda_n = \max W(A) = \max_{0 \neq \phi \in \mathbb{C}^n} \frac{A\phi \cdot \phi}{\phi \cdot \phi},
$$

respectively.
**Theorem 2.3 (Minimax Principle).** Let $\mathcal{A}$ be a closed, symmetric, and semibounded form and $\mathcal{A}$ its associated operator. Then the discrete point spectrum $\sigma_d(A)$ including multiplicities is enumerated by

$$
\lambda_k^-(A) = \min_{\dim U = k} \max_{0 \neq \phi \in U} \mathcal{R}[u] = \min_{U \subseteq H \text{ } \dim U = k} \max_{0 \neq \phi \in U} \mathcal{R}[u]
$$

starting from the left and by

$$
\lambda_k^+(A) = \max_{\dim U = k} \min_{0 \neq \phi \in U} \mathcal{R}[u] = \max_{U \subseteq D(A) \text{ } \dim U = k} \min_{0 \neq \phi \in U} \mathcal{R}[u]
$$

starting from the right.
Remark 2.4. The situation depicted occurs rather seldom. More often . . .

- The spectrum is unbounded above. ➤ All Laplacians.
- The spectrum consists only of discrete eigenvalues accumulating at infinity. ➤ All operators with compact resolvents, esp. all “our” Laplacians. In this case denote

\[ \lambda_k(A) = \lambda_k^{-}(A) = \min_{U \leq H} \max_{0 \neq \phi \in U \dim U = k} \mathcal{R}[u]. \]

- We are interested in the first two eigenvalues (counting multiplicities). Hence, the following equalities are important.

\[ \lambda_1(A) = \min_{0 \neq \phi \in H} \mathcal{R}[u], \]
\[ \lambda_2(A) = \min_{U \leq H} \max_{0 \neq \phi \in U \dim U = 2} \mathcal{R}[u]. \]
2.3 Monotonicity

Notation 2.5. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two forms and $A_1$ and $A_2$ be the associated operators.

- We write $\mathcal{A}_1 \leq \mathcal{A}_2$ if $D(\mathcal{A}_1) \supseteq D(\mathcal{A}_2)$ and $A_1[u] \leq A_2[u]$ for all $u \in D(\mathcal{A}_2)$.

- By abuse of notation we write $\mathcal{A}_1 \leq \mathcal{A}_2$ instead of $A_1 \leq A_2$.

Corollary 2.6 (Monotonicity). Let $\mathcal{A}_1 \leq \mathcal{A}_2$ then $\lambda_k^-(A_1) \leq \lambda_k^-(A_2)$.

Proof. By definition, $\mathcal{A}_1 \leq \mathcal{A}_2$ implies $U \subseteq D(\mathcal{A}_2) \subseteq D(\mathcal{A}_1)$. Hence, the minimum for $\lambda_k^-(A_1)$ is found in a set of subspaces that is at least equal to the set of subspaces $\lambda_k^-(A_2)$ is a minimum on.
Example 2.7 (Laplacians). Since

\[ H^1_0(\Omega) \leq H^1_{\text{per}}(\Omega) \leq H^1(\Omega), \]

for an appropriate set \( \Omega \), we have

\[ -\Delta^\Omega_N \leq -\Delta^\Omega_{\text{per}} \leq -\Delta^\Omega_D. \]

Hence, we get the monotonicity of the eigenvalues

\[ \lambda_k^- (-\Delta^\Omega_N) \leq \lambda_k^- (-\Delta^\Omega_{\text{per}}) \leq \lambda_k^- (-\Delta^\Omega_D). \]
Story from the end

➤ Linear Algebra: Selfadjoint Operators are Diagonal matrices.
3 Representation of Selfadjoint Operators

3.1 Functional Calculus

Notation 3.1 (Multiplication Operators). Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $a \in C(\Omega, \mathbb{R})$ a continuous function hereon. On the space $D(M_a)$, given by

$$D(M_a) := \left\{ v \in L^2(\Omega) \mid a \cdot v \in L^2(\Omega) \right\},$$

we can define an operator by

$$M_a v := a \cdot v.$$

Hence, $M_a v (\xi) = a(\xi) v(\xi)$ for $v \in D(M_a)$. Such an operator is called a Multiplication Operator.
Multiplication Operators generalize the notion of diagonal matrix. They have similar properties:

- For all \( \lambda \in \mathbb{C} \), the translated operator is given by
  \[
  \lambda - M_a = M_{\lambda - a}.
  \]

- The spectrum is given by the image of \( a \):
  \[
  \sigma(M_a) = \{ a(\xi) | \xi \in \Omega \}.
  \]

- \((\lambda - M_a)^{-1} = M_{(\lambda - a)^{-1}}\).
Every selfadjoint operator is actually a multiplication operator.
Theorem 3.2 (Functional Calculus). Let $A$ be a selfadjoint operator on the Hilbert space $L$. Then $A$ is unitary equivalent to a Multiplication Operator $M_a$, i.e. there exists a measure space $(\Omega, \mathcal{B}, \mu)$, an unitary mapping $U : L \to L^2(\Omega, \mu)$, and a measurable function $a : \Omega \to \mathbb{R}$.

Let $D(M_a) := \{ v \in L^2(\Omega) \mid a \cdot v \in L^2(\Omega) \}$ and $M_a v := a \cdot v$ for $v \in D(M_a)$. Then the following diagram commutes:

In words:

$$A = U^* M_a U \text{ on } D(A) = U^* D(M_a)$$

or equivalently for $v \in D(M_a)$

$$a \cdot v = (U A U^*) v \text{ a.e. in } \Omega.$$
Example 3.3. 1. If $A$ is a selfadjoint operator on $\mathbb{C}^n$ there exist $a_1, \ldots, a_n \in \mathbb{R}$ such, that

$$A \sim \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix}.$$

Think of $a$ to be $a : \{1, \ldots, n\} \to \mathbb{R}$.

2. Let $A$ be $-\Delta$ in $L^2(\mathbb{R}^n)$ and $U$ be the Fourier transform. Then for $a(\xi) = |\xi|^2$

$$-\Delta u(x) \sim U |\xi|^2 v(\xi),$$

in the sense of

$$-\Delta \sim U M_a$$

$$D(-\Delta) = \{ u \in L \mid v = U u \in L^2(\mathbb{R}^n) \text{ satisfies } av \in L^2(\mathbb{R}^n) \}.$$
Remark 3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

1. Let $A$ be selfadjoint operator on $\mathbb{C}^n$ then define

$$f(A) \underbrace{U}_{\mathbb{R}} = \begin{bmatrix}
    f(a_1) & 0 & \cdots & 0 \\
    0 & f(a_2) & 0 & \vdots \\
    0 & \ddots & 0 \\
    \vdots & 0 & f(a_{n-1}) & 0 \\
    0 & \cdots & 0 & f(a_n)
\end{bmatrix}.$$

2. Let $A$ be $-\Delta$ in $L^2(\mathbb{R}^n)$. Then for $a(\xi) = |\xi|^2$,

$$e^{\Delta t} u(x) \underbrace{U}_{\mathbb{R}} \approx e^{-|\xi|^2 t} v(\xi)$$

defines the fundamental solution of the heat equation, i.e. the solution of $u_t = \Delta u$ with initial value $u_0 \in L^2(\mathbb{R}^n)$ is given by $u(t) = e^{\Delta t} u_0$. 
Let $A \in \text{Lin}(L)$ then

$$A^0 = \text{Id}, A, A^2, \ldots, A^k, \ldots \in \text{Lin}(L),$$

for $k \in \mathbb{N}$ are well defined. Hence, a linear combination like

$$p(A) = \sum_{k=0}^{\text{grad } p} p_k A^k \in \text{Lin}(L)$$

is well defined for every polynomial $p(X) = \sum_{k=0}^{\text{grad } p} p_k X^k \in \mathbb{C}[X]$.

For example $p(X) = X^3 + i$ yields the operator $p(A) = A^3 + i \text{Id}$, or short $A^3 + i$, resp.

Remark 3.5. Since $A$ is selfadjoint, $\|A\| = \sup \sigma(A) = \| \text{Id} \|_{C(\sigma(A))}$.

- For $p \in \mathbb{R}[X]$ the operator $p(A)$ is selfadjoint.
- For $p \in \mathbb{C}[X]$ the operator $p(A)$ is normal.
Theorem 3.6 (Functional Calculus). For each selfadjoint operator $A \in \text{Lin}(L)$ there exists a unique Functional Calculus $\Phi : \mathbb{C}[X] \to \text{Lin}(L)$, i.e.

**Standardization and Normalization:** $\Phi(1) = 1$ and $\Phi(\text{Id}) = A$.

**Algebra Homomorphism:** $\Phi(\alpha p + q) = \alpha \Phi(p) + \Phi(q)$ and $\Phi(pq) = \Phi(p) \Phi(q)$ for $\alpha \in \mathbb{C}$ and $p, q \in \mathbb{C}[X]$.

**Involutive:** $\Phi(p^\ast) = \Phi(p)^\ast$ for $p \in \mathbb{C}[X]$.

**Isometry and Continuity:** $\|\Phi(p)\| = \|p\|_{C(\sigma(A))}$ for $p \in \mathbb{C}[X]$.

Hence, a Functional Calculus is a continuous, $1-1$, $A$-normalized algebra homomorphism between the $\mathbb{C}$-algebras $$(\mathbb{C}[X], +, \cdot, \cdot^\ast, 0, 1)$$ and $$(\text{Lin}(L), +, \circ, \cdot^\ast, 0, 1 = \text{Id})$$ or short ...

$$\mathbb{C}[A] \leq \text{Lin}(L) \text{ has all the properties of } \mathbb{C}[X] \leq C_b(\sigma(A)).$$
Remark 3.7 (Extension of the Functional Calculus).

- Density \( C[X] \) in \( C(\sigma(A)) \) yields Functional Calculus: For \( f \in C(\sigma(A)) \) define
  \[
f(A) = \lim p_k(A) \quad \text{in Lin}(L).
\]

- Density of \( C(\sigma(A)) \) in \( L^\infty(\sigma(A)) \), the space of bounded, measurable functions, w.r.t. pointwise limits of uniformly bounded sequences. For \( f \in L^\infty(\sigma(A)) \) define
  \[
f(A)x = \lim f_k(A)x \quad \text{in } L.
\]
Theorem 3.8 (Functional Calculus).

For each bounded, selfadjoint operator $A$ there exists a unique Functional Calculus $\Phi : L^\infty(\sigma(A)) \to \text{Lin}(L)$, i.e.

**Standardization and Normalization:** $\Phi(1) = 1$ and $\Phi(\text{Id}) = A$.

**Algebra Homomorphism:** $\Phi(\alpha f + g) = \alpha \Phi(f) + \Phi(g)$ and $\Phi(fg) = \Phi(f)\Phi(g)$ for $\alpha \in \mathbb{C}$ and $f, g \in L^\infty(\sigma(A))$.

**Involutive:** $\Phi(\overline{f}) = \Phi(f)^*$ for $f \in L^\infty(\sigma(A))$.

**Continuity:** $\|\Phi(f)\| = \|f\|_{L^\infty(\sigma(A))}$ for $f \in \mathbb{C}[X]$.

So, for short, we get an embedding of $L^\infty(\sigma(A))$ into $\text{Lin}(L)$, which respects all the structures of the commutative $\mathbb{C}$-algebra $L^\infty(\sigma(A))$. 
Example 3.9 (Wave Equation). Let $A$ be a non-negative, bounded operator. Then the solution of the wave-type equation

$$
\begin{align*}
  u_{tt} + Au &= 0 \quad \text{in } L \times \mathbb{R}, \\
  u(0) &= u_0 \quad \text{in } L, \quad \text{and} \\
  u_t(0) &= v_0 \quad \text{in } L
\end{align*}
$$

is given by

$$
u(t) = \cos(t) u_0 + \sin(t) v_0,$$

where the **Cosine-Family** $\cos = \{\cos(t)\}_{t \in \mathbb{R}}$ and the **Sine-Family** $\sin = \{\sin(t)\}_{t \in \mathbb{R}}$ are given by

$$
\begin{align*}
  \cos(t) &= \cos\left(\sqrt{A} \ t\right) \quad \text{and} \quad \sin(t) &= \int_0^t \cos(s) \, ds.
\end{align*}
$$
Proposition 3.10.

Let $A$ be a bounded, selfadjoint operator. Then the associated cosine family $\text{Cos}$ and sine family $\text{Sin}$ have the following properties:

1. $\text{Sin}(t)$ and $\text{Cos}(t)$ are bounded and selfadjoint operators.

2. $\text{Sin} : \mathbb{R} \to \text{Lin}(L)$ and $\text{Cos} : \mathbb{R} \to \text{Lin}(L)$ are differentiable. More precisely:

\[
\frac{d}{dt} \text{Sin}(t) = \text{Cos}(t) \quad \text{for all } t \in \mathbb{R},
\]

\[
\frac{d}{dt} \text{Cos}(t) = A \text{Sin}(t) \quad \text{for all } t \in \mathbb{R}.
\]
For each (measurable) set $U \subseteq \mathbb{R}$, the characteristic function

$$1_U(x) = \begin{cases} 
1 & \text{for } x \in U, \text{ and} \\
0 & \text{otherwise},
\end{cases}$$

can be used to define a projection

$$E_A(U) := 1_U(A) \in \text{Lin}(L).$$

Proposition 3.11. $E_A(U)$ is a orthogonal projection for each $U \subseteq \mathbb{R}$.

Proof. $E_A(U)$ is a projection due to the properties of the functional calculus:

$$E_A(U)E_A(U) = 1_U(A)1_U(A) = (1_U1_U)(A) = 1_U(A) = E_A(U).$$

It is selfadjoint by $E_A(U)^* = (1_U)(A)^* = 1_U(A) = 1_U(A) = E_A(U)$. Selfadointness is equivalent to orthogonal decomposition of the image and the kernel.
Theorem 3.12 (Spectral Resolution relative to $A$). Let $A \in \text{Lin}(L)$ be selfadjoint. Then $E_A : \mathcal{B} (\sigma(A)) \to \text{Lin}(L)$ is a spectral measure, i.e.

1. $E_A(\emptyset) = 0$ and $E_A(\mathbb{R}) = 1$.

2. Let $U_k \subseteq \mathbb{R}$ be pairwise disjoint, then

$$E_A \left( \bigcup_{k \in \mathbb{N}} U_k \right) = \sum_{k \in \mathbb{N}} E_A(U_k).$$

3. Let $U_1, \ldots, U_N \subseteq \mathbb{R}$ be any sets, then

$$E_A \left( \bigcap_{k=1}^{N} U_k \right) = \prod_{k=1}^{N} E_A(U_k).$$

Since $1_{\sigma(A)} = 1_{\mathbb{R}}$, considered as functions in $\mathcal{L}^\infty(\sigma(A))$, $\text{supp } E_A = \sigma(A)$. Hence, $E_A$ is a $\text{Lin}(L)$ valued measure with compact support $\sigma(A)$. 

Theorem 3.13 (Spectral Resolution and Functional Calculus are consistent).

Let \( A \in \text{Lin}(L) \) be selfadjoint, \( E_A : \mathcal{B}(\sigma(A)) \to \text{Lin}(L) \) its spectral measure. For \( f \in \mathcal{L}^\infty(\sigma(A)) \) and \( u, v \in L \)

\[
\langle f(A)u, v \rangle = \int f(\lambda) \, d\langle E_A(\lambda)u, v \rangle.
\]

Remark 3.14. The integral \( \int f(\lambda) \, dE_A(\lambda) \) is constructed similar to the Lebesgue-Integral: Integral for simple functions; integration is continuous; simple functions are dense in \( \mathcal{L}^\infty(\sigma(A)) \).
Remark 3.15.

- Decomposition of a normal operator $A = S_1 + iS_2$ into two commuting selfadjoint operators generalizes the Functional Calculus and the Spectral Resolution to normal, bounded operators.

- Every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is in the resolvent set $\rho(A)$ of a not necessarily bounded, but selfadjoint, operator $A$. Since $R_\lambda = (\lambda - A)^{-1}$ is a bounded and normal operator, all ideas can be generalized to the unbounded case.
Finally, we are able to state the following:

Let $A$ be a selfadjoint operator, $\mathcal{A}$ be its form with domain $D(A)$. Then the first and second eigenvalue, counting multiplicities, are given by

$$\lambda_1(A) = \min_{0 \neq \phi \in D(A)} \frac{\mathcal{A}[\phi, \phi]}{\langle \phi, \phi \rangle}$$

$$\lambda_2(A) = \min_{U \leq D(A)} \max_{0 \neq \phi \in U, \dim U = 2} \frac{\mathcal{A}[\phi, \phi]}{\langle \phi, \phi \rangle}.$$ 

So it is sufficient to work with the quadratic form domain.

Let $S$ be a selfadjoint and bounded operator. For a continuous function $f$ the operator $f(S)$ is defined.

A selfadjoint and bounded operator $S$ is characterized uniquely by its Spectral Resolution: $S = \int \lambda \, dE_S(\lambda)$. 

Asymptotics of Non-local Spectral Problems (34/65)
4 Structure of the Problem

4.1 Framework of the involved Problems.

The considered GEPs are of the following form:

\[
\int_{\Omega} D \phi \cdot D \psi = \mu \sum_{\alpha} \left( \int_{\gamma(\alpha)} \phi \nu \right) \cdot \left( \int_{\gamma(\alpha)} \psi \nu \right) =: b_\alpha(\phi) \cdot b_\alpha(\psi) =: B[\phi, \psi]
\]

Here \( A \) and \( B \) are well defined forms on the considered \( H^1 \)—spaces and every subspace thereof. Both vanish on constants. Therefore, we identify constant functions with zero.
Reason for the •: Complex scalar product! In $\ell^2$. 
The complete reference cell is given by

\[ Y = (0, \tau_1) \times (0, \tau_2). \]

Removing the tube with smooth boundary \( \gamma \) from the cell yields the reference cell \( Y^* \).

For \( \alpha \in \mathbb{Z}^2 \) the translated cell/boundary/... is given by

\[ Y(\alpha) = Y + (\alpha_1 \tau_1, \alpha_2 \tau_2) \]
\[ Y^*(\alpha) = Y^* + (\alpha_1 \tau_1, \alpha_2 \tau_2) \]
\[ \gamma(\alpha) = \gamma + (\alpha_1 \tau_1, \alpha_2 \tau_2) \]
\[ \ldots \quad = \quad \ldots \]
The considered domains with a finite and an infinite number of tubes are given by

\[ \Omega_n = \Omega \cap (-n\tau_1, (n - 1)\tau_1) \times (-n\tau_2, (n - 1)\tau_2) \]

and

\[ \Omega_\infty = \text{Int} \left( \bigcup_{\alpha \in \mathbb{Z}^2} Y^*(\alpha) \right), \]

respectively.
We use the following boundary conditions on each bounded domain:

- Neumann-Condition.
- $\Omega_n$-Periodicity.
- $\omega$-Periodicity on each cell.
- Dirichlet-Condition on each cell.

In the case of infinite number of tubes we only will pose a bound on the growth of the functions at infinity: $D\phi \in L^2$. These ideas will be modeled through the choice of spaces.
Asymptotic bound on the gradient:

\[
H_\infty := \left( C_c^\infty(\overline{\Omega_\infty}) \right)^{\nabla \cdot H^1} = \left\{ \phi \in L^2_{\text{loc}}(\Omega_\infty) \mid \nabla \phi \in L^2(\Omega_\infty)^2 \right\} / \mathbb{C}.
\]

Neumann-Condition: \( H_n := H^1(\Omega_n) / \mathbb{C} \).

\( \Omega_n \)-Periodicity: \( H_{n,\text{per}} := H^1_{\text{per}}(\Omega_n) / \mathbb{C} \).

\( \omega \)-Periodicity on each cell: Some contain the constants. Hence,

For \( \omega \neq (1, 1) \), \( H_\omega := H^1_{\text{per}}(\omega, Y^*) \).

For \( \omega = (1, 1) \), \( H_\omega := H^1_{\text{per}}(Y^*) / \mathbb{C} \).

Furthermore, we abbreviate \( H_\alpha := H_{\omega^\alpha} \).

Each of this spaces is a Hilbert space with inner product \( \mathcal{A} \).

Here \( n \in \mathbb{N}, \omega \in S_{\mathbb{C}} \times S_{\mathbb{C}}, \) and \( \alpha \in \mathbb{Z}^2 \).
As we will see, on each of these spaces the forms $\mathcal{A}$ and $\mathcal{B}$ and the operator $b$ make sense. Their respective versions will have the following names:

**Asymptotic Problem on** $H_\infty$: $\mathcal{A}_\infty$, $\mathcal{B}_\infty$, and $b_\infty$.

**Neumann-Condition on** $H_n$: $\mathcal{A}_n$, $\mathcal{B}_n$, and $b_n$.

**$\Omega_n$-Periodicity on** $H_{n,\text{per}}$: $\mathcal{A}_{n,\text{per}}$, $\mathcal{B}_{n,\text{per}}$, and $b_{n,\text{per}}$.

**$\omega$-Periodicity on each cell** $H_\omega$ or $H_\alpha$: $\mathcal{A}_\omega$, $\mathcal{B}_\omega$, and $b_\omega$ or $\mathcal{A}_\alpha$, $\mathcal{B}_\alpha$, and $b_\alpha$. 
We will now consider each problem separately:
Rike: Decomposition Theorem
Properties of the Cell Problem: $H_\omega$.

The eigenvalue problem on the cell is of the form: Find $0 \neq \phi_\omega \in H_\omega$ such that

$$A_\omega \phi_\omega = \mu \ B_\omega \phi_\omega \quad \text{in } H_\omega^*.$$ 

By construction, $A$ is bijective, since it actually defines the inner product on $H_\omega$. Hence, the problem is equivalent to

$$\lambda \phi_\omega = \underbrace{(A_\omega)^{-1} \ B_\omega \phi_\omega}_{=: C_\omega} \quad \text{in } H_\omega.$$ 

Thus, to solve our generalized eigenvalue problem, we have to locate the non trivial part

$$\sigma_p(C_\omega) \setminus \{0\}$$

of the point spectrum of the bounded operator $C_\omega$. 

Asymptotics of Non-local Spectral Problems (41/65)
Properties of the Cell Problem: $H_\omega$.

For $r \in \mathbb{C}^2$ we can define an operator $\mathcal{B}_{\omega,\text{red}} : \mathbb{C}^2 \to H_\omega^*$ by $\mathcal{B}_{\omega,\text{red}} r = r \cdot b_\omega$, or equivalently, for $\phi \in H_\omega$

$$\mathcal{B}_{\omega,\text{red}} r (\phi) = r \cdot \int_\gamma \phi \nu \in \mathbb{C},$$

which enables us to split $\mathcal{B}_\omega$ into $b_\omega$ and $\mathcal{B}_{\omega,\text{red}}$:

$$\mathcal{B}_\omega [\phi_\omega, \psi_\omega] = b_\omega (\phi_\omega) \cdot b_\omega (\psi_\omega)$$

$$= (\mathcal{B}_{\omega,\text{red}} b_\omega (\phi_\omega)) (\psi_\omega).$$

For a eigenfunction $\phi_\omega$ of $\mathcal{C}_\omega$ and $r_\omega := b_\omega (\phi_\omega)$ we get

$$\lambda r_\omega = \lambda b_\omega (\phi_\omega)$$

$$= b_\omega (\mathcal{A}_\omega^{-1} \mathcal{B}_{\omega,\text{red}}) (b_\omega (\phi_\omega))$$

$$= \underbrace{b_\omega (\mathcal{B}_{\omega,\text{red}} r_\omega)}_{=: S_\omega} = \underbrace{(b_\omega \mathcal{A}_\omega^{-1} \mathcal{B}_{\omega,\text{red}})}_{=: S_\omega} r_\omega$$
The main idea now is: The following problems describe the same set of eigenvalues \( \mu \neq 0 \) including all multiplicities, where \( \lambda = \frac{1}{\mu} \).

1. Find \( 0 \neq \phi \in H_\omega \), such that

\[
A_\omega \phi = \mu B_\omega \phi \quad \text{in } H_\omega^*.
\]

2. Find \( 0 \neq \phi \in H_\omega \), such that

\[
\lambda \phi = C_\omega \phi \quad \text{in } H_\omega.
\]

3. Find \( 0 \neq r \in \mathbb{C}^2 \), such that

\[
\lambda r = S_\omega r \quad \text{in } \mathbb{C}^2.
\]

So we reduced the complexity of problem (1) to the simpler form of problem (3).
Proposition 4.1. The operators $\{b_\omega\}_\omega$, given by

$$b_\omega : H_\omega \to \mathbb{C}^2 : \phi \mapsto \int_\gamma \phi \nu,$$

are linear, continuous, and surjective. Actually, it can be shown, that $b_\omega$ is already surjective if restricted to

$$H_0 := \{ \phi \in H^1(Y^*) \mid \phi = 0 \text{ on } \partial Y \}.$$

Proposition 4.2. $A_\omega$ defines an inner product on $H_\omega$, which is equivalent to the standard inner product.
Proposition 4.3. The operators \( \{S_\omega\}_\omega \) are selfadjoint and satisfy \( \bar{\lambda} \geq S_\omega \geq \lambda \), i.e.

\[
\bar{\lambda} r \cdot r \geq S_\omega r \cdot r \geq \lambda r \cdot r
\]

for all \( r \in \mathbb{C}^2 \) and \( \omega \in S_C \times S_C \), where \( 0 < \lambda \leq \bar{\lambda} < \infty \) can be chosen independent of \( \omega \).

Proof. Let \( r_\omega \in \mathbb{C}^2 \) and define \( \phi_\omega := A_\omega^{-1} B_{\omega,\text{red}} r_\omega \in H_\omega \). We get

\[
\mathbb{R} \ni A_\omega[\phi_\omega, \phi_\omega] = A_\omega (A_\omega^{-1} B_{\omega,\text{red}} r_\omega) (A_\omega^{-1} B_{\omega,\text{red}} r_\omega) = B_{\omega,\text{red}} r_\omega (A_\omega^{-1} B_{\omega,\text{red}} r_\omega) = r_\omega \cdot b (A_\omega^{-1} B_{\omega,\text{red}} r_\omega) = r_\omega \cdot S_\omega r_\omega.
\]

Hence, \( S_\omega r_\omega \cdot r_\omega = \overline{r_\omega \cdot S_\omega r_\omega} = r_\omega \cdot S_\omega r_\omega \) proves symmetry of all \( S_\omega \) and \( A_\omega[\phi_\omega, \phi_\omega] \geq 0 \) implies non-negativity.
Properties of the Cell Problem: $H_\omega$.

Idea of the proof of the uniform bounds:

1. The upper bound is given by the largest eigenvalue of $S_\omega$. But this eigenvalue is also the largest eigenvalue of $C_\omega$. Therefore it is given by the minimax principle as maximum over $H_\omega$. Since $H_\omega \leq H^1(Y^*)/C$ the largest eigenvalues are uniformly bounded by the largest eigenvalue of $A^{-1}B$ on $H^1(Y^*)/C$. This proves the upper bound.

2. The second largest eigenvalue of $S_\omega$, which b.t.w. is also the smallest eigenvalue, can be described as the second largest eigenvalue of $C_\omega$:

$$
\lambda^+_2(S_\omega) = \lambda^+_2(C_\omega) = \max_{U \leq H_\omega} \min_{\dim U = 2} \frac{\langle C_\omega \phi, \phi \rangle_{H_\omega}}{\langle \phi, \phi \rangle_{H_\omega}}.
$$

Let

$$
H_0 := \left\{ \phi \in H^1(Y^*) \mid \phi = 0 \text{ on } \partial Y \right\}.
$$
This space can be considered as $H_0 \leq H_\omega$. Hence, monotonicity in the spaces gives

$$\lambda_2^+(S_\omega) = \lambda_2^+(C_\omega) \geq \max_{U \leq H_0} \min_{\dim U = 2} \frac{\langle C_\omega \phi, \phi \rangle_{H_\omega}}{\langle \phi, \phi \rangle_{H_\omega}},$$

i.e.

$$\lambda_2^+(C_\omega) \geq \lambda_2^+(C|_{H_0}) > 0.$$

This proves the lower bound, independent of $\omega$.

Let $\mu_k(S_\omega) := \frac{1}{\lambda_k(S_\omega)}$, and let $r_\omega^1$ and $r_\omega^2$ be the two eigenvectors of $S_\omega$. Hence

$$\mu_k S_\omega r_\omega^k = r_\omega^k \quad \text{for } k, l = 1, 2.$$
Properties of the Cell Problem: $H_\omega$.

We can assume

$$r^k_\omega \cdot r^l_\omega = \mu_l (S_\omega) \delta_{kl} \quad \text{for } k, l = 1, 2.$$ 

Let

$$\phi^k_\omega := A^{-1}_\omega B_{\omega, \text{red}} r^k_\omega \quad \text{for } k = 1, 2.$$ 

Then for $k = 1, 2$, we get $A_\omega \phi^k_\omega = B_{\omega, \text{red}} r^k_\omega$, and therefore for $\psi \in H_\omega$

\[
A_\omega [\phi^k_\omega, \psi] = r^k_\omega \cdot b(\psi) = \mu_k S_\omega r^k_\omega \cdot b(\psi) = \mu_k B_{\omega} [\phi^k_\omega, \psi].
\]

Hence, $A_\omega \phi^k_\omega = \mu_k B_{\omega} \phi^k_\omega$ for $k, l = 1, 2$. Furthermore,

\[
A_\omega [\phi^k_\omega, \phi^k_\omega] = \mu_k B_{\omega} [\phi^k_\omega, \phi^k_\omega] = \mu_k b(A^{-1}_\omega B_{\omega, \text{red}} r^k_\omega) \cdot b(A^{-1}_\omega B_{\omega, \text{red}} r^l_\omega) = \mu_k \lambda_k \lambda_l r^k_\omega \cdot r^l_\omega = \mu_k \lambda_k \lambda_l \mu_l \delta_{kl} = \delta_{kl}.
\]
Properties of the Cell Problem: $H_\omega$.

Let now $\mu \neq 0$ be a generalized eigenvalue with eigenfunction $\phi \in H_\omega$, i.e. $A_\omega \phi = \mu_k B_\phi$. Then define $r := b_\omega(\phi)$. Since $0 \neq \phi = \mu_k (A^{-1}_\omega B_{\omega,\text{red}}) r$, the vector $r$ must non trivial. Furthermore, $r = \mu_k b_\omega (A^{-1}_\omega B_{\omega,\text{red}}) r = \mu_k S_\omega r$ shows, that $(r, \mu_k)$ must be a characteristic pair of $S_\omega$. Hence $r$ is linear dependent on $r^1_\omega$ and $r^2_\omega$ and equals $\mu_1$ or $\mu_2$.

**Theorem 4.4.** For all $\omega \in S_C \times S_C$ the complete set of generalized eigenvalues is given by

$$0 < \mu \leq \mu_1(\omega) \leq \mu_2(\omega) \leq \overline{\mu} < \infty.$$  

The corresponding eigenfunctions $\phi^1_\omega$ and $\phi^2_\omega \in H_\omega$ can be chosen such, that for $k, l = 1, 2$

$$\langle \phi^k_\omega, \phi^l_\omega \rangle_{H_\omega} = A[\phi^k_\omega, \phi^l_\omega] = \delta_{kl} \quad \text{and} \quad A\phi^k_\omega = \mu_k \ B\phi^k_\omega.$$
The bigger problem
The considered domain with a finite number of tubes is given by

\[ \Omega_n = \Omega \cap (-n\tau_1, (n - 1)\tau_1) \times (-n\tau_2, (n - 1)\tau_2) \]

The domain \( \Omega_2 \).
Properties of the periodic Problem: $H_{n,\text{per}}$.

Let $\omega = (e^{\frac{2\pi i}{m}}, e^{\frac{2\pi i}{n}})$ and denote for $\alpha \in \mathbb{Z}^2$

$$\omega^\alpha = (e^{\frac{2\pi i\alpha_1}{m}}, e^{\frac{2\pi i\alpha_2}{n}}).$$

Then the following **Orthogonal Decomposition Theorem** holds true:

- $H_{n,\text{per}} = \bigoplus_{\alpha \in \{-n,\ldots,n-1\}^2} H_\alpha$
- For $\alpha \neq \beta$ and $u_\alpha \in H_\alpha$, $u_\beta \in H_\beta$:
  $$u_\alpha \perp_{L^2} u_\beta, \quad u_\alpha \perp_{A_{n,\text{per}}} u_\beta, \text{ and } u_\alpha \perp_{B_{n,\text{per}}} u_\beta.$$
- For $u_\alpha \in H_\alpha$
  $$A_{n,\text{per}} u_\alpha \in H_\alpha^* \text{ and } B_{n,\text{per}} u_\alpha \in H_\alpha^*.$$

Hence, $A_{n,\text{per}}$ and $B_{n,\text{per}}$ can be considered as

$$A_{n,\text{per}} = \bigoplus A_\alpha \quad \text{and} \quad B_{n,\text{per}} = \bigoplus B_\alpha.$$
Properties of the periodic Problem: $H_{n,\text{per}}$.

Therefore we get the following formulations of the eigenvalue problems:

\[
\mathcal{A}_{n,\text{per}} \phi_n = \mu \mathcal{B}_{n,\text{per}} \phi_n \quad \text{in } H_{n,\text{per}}^*,
\]

\[
\lambda \phi_n = \mathcal{C}_{n,\text{per}} \phi_n \quad \text{in } H_{n,\text{per}},
\]

\[
\lambda r = \mathcal{S}_{n,\text{per}} r \quad \text{in } \ell^2_n(C^2).
\]

Here, the space $\ell^2_n(C^2)$ is given by

\[
\ell^2_n(C^2) := \ell^2([-n, \ldots, n-1]^2; C^2).
\]

Let $L_n := 4n^2$, i.e. the number of tubes, then $\ell^2_n(C^2)$ is just a strange way, to describe the space $C^{2L_n}$ with the standard inner product. For $r, s \in \ell^2_n(C^2)$ the product is denoted by $r \cdot s$. The decomposition theorem provides us with compatible decompositions. We can consider the problems in each space $H_\alpha$. Reformulating this approach and adding up all the results, we get the main theorem.
Properties of the periodic Problem: $H_{n,\text{per}}$.

**Theorem 4.5.** For every $n \in \mathbb{N}$ each generalized eigenvalue problem

$$
\int_{\Omega} \nabla \phi \cdot \nabla \psi = \mu \sum_{\alpha} \left( \int_{\gamma(\alpha)} \phi \nu \right) \cdot \left( \int_{\gamma(\alpha)} \psi \nu \right), 
$$

(GEP)

posed in $H^1_{\text{per}}(\Omega_n)/\mathbb{C}$, has exactly $8n^2 = 2 \times \text{Number of holes solutions}$, counting all multiplicities. They satisfy independently of $n$

$$
0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_{8n^2-1} \leq \mu_{8n^2} \leq \bar{\mu} < \infty.
$$

The corresponding eigenfunctions $\phi^k$, for $k, l = 1, \ldots, 8n^2$, chosen such, that

$$
\langle \phi^k, \phi^l \rangle_{H_{n,\text{per}}} = \mathcal{A}[\phi^k, \phi^l] = \delta_{kl} \quad \text{and} \quad \mathcal{A}_{n,\text{per}} \phi^k = \mu_k \quad \mathcal{B}_{n,\text{per}} \phi^k,
$$

are called **Bloch Waves** (of the GEP).
Like in the periodic case, the eigenvalue problems

\[ A_n \phi_n = \mu B_n \phi_n \quad \text{in} \quad H_n^*, \]
\[ \lambda \phi_n = C_n \phi_n \quad \text{in} \quad H_n, \]
\[ \lambda r = S_n r \quad \text{in} \quad \ell_n^2(\mathbb{C}^2), \]

are considered.

Contrary to the periodic case, we lack a decomposition theorem. Nevertheless, every \( S_n \) has the following properties, which can be proved similarly to the case \( S_\omega \).

**Proposition 4.6.** For all \( n \in \mathbb{N} \) the operators \( S_n \) are selfadjoint and uniformly strictly positive:

\[ S_n r_n \cdot r_n \geq \lambda r_n \cdot r_n \quad \text{for all} \quad r_n \in \ell_n^2(\mathbb{C}^2) \quad \text{and} \quad n \in \mathbb{N}. \]
Properties of the Problem: \( H_n \).

Since \( S_n \) is a selfadjoint operator, it possesses a system \( \{ r_n^{(k)} \} \) of eigenfunctions. These can be transferred into eigenfunction of \( C_n \) like in the periodic case, using

\[ \phi_n^{(k)} = A_n^{-1} B_{n,\text{red}} r_n^{(k)} \]

Finally we get

**Theorem 4.7.** For all \( n \in \mathbb{N} \) each generalized eigenvalue problem

\[
\int_\Omega D \phi \cdot D \psi = \mu \sum_\alpha \left( \int_{\gamma(\alpha)} \phi \nu \right) \cdot \left( \int_{\gamma(\alpha)} \psi \nu \right) \text{ for all } \psi \in H_n,
\]

posed in \( H_n \), has exactly \( 8n^2 = 2 \times \text{Number of holes solutions, counting all multiplicities.} \) They satisfy independently of \( n \)

\[
0 < \mu_0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_{8n^2-1} \leq \mu_{8n^2}.
\]

The corresponding eigenfunctions \( \phi_n^{(k)} \), for \( k, l = 1, \ldots, 8n^2 \), can be chosen orthonormal in \( H_n \).
Properties of the Asymptotic Problem: \( H_\infty \).

To prove, that similar ideas also work in this case, the infinite number of tubes case needs some consideration.

**Proposition 4.8.** Denote \( \ell^2(C^2) := \ell^2(\mathbb{Z}^2; C^2) \). The operator

\[
b_\infty : H_\infty \to \ell^2(C^2) : \phi \mapsto \left( \int_{\gamma(\alpha)} \phi \nu \right)_{\alpha \in \mathbb{Z}^2}
\]

is well-defined and continuous.

**Proof.** For \( \phi \in H_\infty \) we have on each cell

\[
\left| \int_{\gamma(\alpha)} \phi \nu \right|^2 \leq \| \phi \|_{L^2(\gamma(\alpha))}^2 \| \nu \|_{L^2(\gamma(\alpha))}^2 \\
\leq c \| D \phi \|_{L^2(\gamma(\alpha))}^2
\]

since on \( H^1(Y^*(\alpha))/\mathbb{C} \) the norm is given by \( \| D \cdot \|_{L^2(Y^*(\alpha))} \).
This constant is independent of $\alpha$. Thus,

$$\left| b_\infty(\phi) \right|^2 = \sum_\alpha \left| \int_{\gamma(\alpha)} \phi \nu \right|^2 \leq c \sum_\alpha \| D\phi \|_{L^2(Y^*(\alpha))}^2 = c \sum_\alpha \| D\phi \|_{L^2(\Omega_\infty)}^2 = c \| \phi \|_{H^\infty}^2.$$ 

Hence, $b_\infty$ is well defined and continuous.

The continuity of $b_\infty$ yields the continuity of $B_\infty$ and of $B_\infty,\text{red}$. So we obtain

**Proposition 4.9.** The operator

$$S_\infty : \ell^2(C^2) \rightarrow \ell^2(C^2) : r \mapsto b_\infty A_\infty^{-1} B_\infty,\text{red} r$$

is well-defined, continuous, and selfadjoint.
Remark 4.10 (Extension of $S_n$ to $\ell^2(\mathbb{C}^2)$). Every operator $S_n$ can be considered as an operator on $\ell^2(\mathbb{C}^2)$ by restriction and extension:

- An element $r_n \in \ell^2_n(\mathbb{C}^2)$ can be extended by zero to an element $r_\infty \in \ell^2(\mathbb{C}^2)$. This defines an embedding $\ell^2_n(\mathbb{C}^2) \leq \ell^2(\mathbb{C}^2)$.
- An element $r_\infty \in \ell^2(\mathbb{C}^2)$ can be restricted to an element $r_n \in \ell^2_n(\mathbb{C}^2)$.
- Therefore, $S_n r_\infty \in \ell^2(\mathbb{C}^2)$ is well defined for $r_\infty \in \ell^2(\mathbb{C}^2)$.

Theorem 4.11 (Extension of $H_n$ to $H_\infty$). There exists a sequence of operators $E_n \in \text{Lin}(H_n, H_\infty)$, that is uniformly continuous: For $\phi \in H_n$

\[
E_n \phi(x) = \phi(x) \text{ for } x \in \Omega_n,
\]

\[
\|E_n \phi\|_{H_\infty} \leq c \|\phi\|_{H_n}, \text{ and}
\]

\[
E_n \phi = 0 \text{ on } \gamma(\alpha) \text{ if } \alpha \notin \{-n, \ldots, n - 1\}^2,
\]

where $c$ is independent of $n$ and $\phi$. 
Now we are able to state the first asymptotic result.

**Theorem 4.12.** For all \( r \in \ell^2(\mathbb{C}^2) \) we have

\[
S_n r \rightarrow S r \quad \text{in } \ell^2(\mathbb{C}^2) \quad \text{for } n \rightarrow \infty.
\]

Especially,

\[
S r \cdot r \geq \lambda r \cdot r \quad \text{for all } r \in \ell^2(\mathbb{C}^2),
\]

where \( \lambda \) is the uniform positivity constant of the operators \( S_n \).

Let \( \text{Sin}_n \) and \( \text{Cos}_n \) be the sine and cosine families generated by \( S_n \), and \( \text{Sin}_\infty \) and \( \text{Cos}_\infty \) be the families generated by \( S_\infty \). All these operator families can be considered as families of operators in \( \ell^2(\mathbb{C}^2) \).
For example, the cosine family is given by

$$\text{Cos}_n : \mathbb{R} \to \text{Lin} \left( \ell^2(\mathbb{C}^2) \right) : t \mapsto \cos(\sqrt{S_n} t)$$

For \( r \in \ell^2(\mathbb{C}^2) \), we have \( \text{Cos}_n r : \mathbb{R} \to \ell^2(\mathbb{C}^2) : t \mapsto \cos(\sqrt{S_n} t) r \).

For \( r, s \in \ell^2(\mathbb{C}^2) \), \( \text{Cos}_n r \cdot s : \mathbb{R} \to \mathbb{C} : t \mapsto \cos(\sqrt{S_n} t) r \cdot s \).

**Theorem 4.13.** All operators are considered as operators on \( \ell^2(\mathbb{C}^2) \). Then for \( r, s \in \ell^2(\mathbb{C}^2) \)

$$\text{Sin}_n r \cdot s \xrightarrow{\ast} \text{Sin}_\infty r \cdot s \quad \text{in} \ L^\infty(\mathbb{R}) \quad \text{for } n \to \infty \quad \text{and}$$

$$\text{Cos}_n r \cdot s \xrightarrow{\ast} \text{Cos}_\infty r \cdot s \quad \text{in} \ L^\infty(\mathbb{R}) \quad \text{for } n \to \infty.$$
Finally, we are able to state and prove the main theorem.

**Theorem 4.14.** or \( r, s \in \ell^2(\mathbb{C}^2) \)

\[
E\sqrt{S_n} r \cdot s \quad \rightarrow \quad E\sqrt{S_n} r \cdot s \quad \text{in} \quad S(\mathbb{R})^* \quad \text{for} \quad n \rightarrow \infty.
\]

**Proof.** For \( r, s \in \ell^2(\mathbb{C}^2) \) the convergence

\[
\Cos_n r \cdot s \quad \overset{*}{\longrightarrow} \quad \Cos_{\infty} r \cdot s \quad \text{in} \quad L^\infty(\mathbb{R}) \quad \text{for} \quad n \rightarrow \infty
\]

implies

\[
\Cos_n r \cdot s \quad \rightarrow \quad \Cos_{\infty} r \cdot s \quad \text{in} \quad S(\mathbb{R})^* \quad \text{for} \quad n \rightarrow \infty.
\]
By consistency of Functional Calculus and Spectral Resolution we have

\[
\cos_n(t) r \cdot s = \int_\lambda^\infty \cos(\sqrt{\lambda} t) \, dE_{\text{Sn}}(\lambda) r \cdot s
\]

\[
= \int_{\sqrt{\lambda}}^\infty \cos(\lambda t) \, dE_{\sqrt{\text{Sn}}}(\lambda) r \cdot s
\]

\[
= \frac{1}{2} \int_{\sqrt{\lambda}}^\infty e^{i\lambda t} + e^{-i\lambda t} \, dE_{\sqrt{\text{Sn}}}(\lambda) r \cdot s
\]

\[
= \frac{1}{2} \int_{\sqrt{\lambda}}^\infty e^{i\lambda t} \, d\left(E_{\sqrt{\text{Sn}}}(\lambda) - E_{\sqrt{\text{Sn}}}(-\lambda)\right) r \cdot s
\]

\[
= \mu_n(\lambda)
\]
Similarly, we define the measure $\mu_\infty$, and obtain

\[
\cos_n(t) r \cdot s = \frac{1}{2} \int e^{i\lambda t} d\mu_n(\lambda) \quad \text{and}
\]

\[
\cos_\infty(t) r \cdot s = \frac{1}{2} \int e^{i\lambda t} d\mu_\infty(\lambda).
\]

Therefore, convergence of $\cos_n r \cdot s$ to $\cos_\infty r \cdot s$ in $\mathcal{S}(\mathbb{R})^*$ is equivalent to

\[
\mathcal{F}\mu_n \longrightarrow \mathcal{F}\mu_\infty \quad \text{in } \mathcal{S}(\mathbb{R})^* \quad \text{as } n \to \infty.
\]

Inverting the Fourier transform in $\mathcal{S}(\mathbb{R})^*$ proves

\[
\mu_n \longrightarrow \mu_\infty \quad \text{in } \mathcal{S}(\mathbb{R})^* \quad \text{as } n \to \infty.
\]
Since, in the sense of measures,

\[
\mu_n(\lambda) = \begin{cases} 
E \sqrt{S_n}(\lambda) r \cdot s & \text{for } \lambda \geq \lambda \\
0 & -\lambda < \lambda < \lambda \\
-E \sqrt{S_n}(-\lambda) r \cdot s & \text{for } \lambda \leq -\lambda 
\end{cases}
\]

we can reformulate this result and get

\[
E \sqrt{S_n} r \cdot s \quad \rightarrow \quad E \sqrt{S_\infty} r \cdot s \quad \text{in } \mathcal{S}(\mathbb{R})^* \quad \text{as } n \to \infty.
\]
References


