2. The Structure of Graphs

- Cut-vertices, bridges, and blocks
- The reconstruction problem
- Graphical parameters and graphical properties

2.1 Cut-vertices, Bridges and Blocks

A vertex \( v \) of a graph \( G \) is called a cut-vertex of \( G \) if \( k(G-v) > k(G) \).

**Theorem 2.1:** A vertex \( v \) of a graph \( G \) is a cut-vertex of \( G \) iff there exist vertices \( u \) and \( w \) \((u, w \in V)\) such that \( v \) is on every \( u-w \) path of \( G \).

**Proof.**

It suffices to prove the theorem for connected graphs.

A: \( \Leftarrow \)

Let \( v \) be a cut-vertex of \( G \). Then the graph \( G-v \) is disconnected.

If \( u \) and \( w \) are vertices in different components of \( G-v \), then there are no \( u-w \) paths in \( G-v \).

However, since \( G \) is connected, there are \( u-w \) paths in \( G \).

Therefore, every \( u-w \) path of \( G \) contains \( v \).

B: \( \Rightarrow \)

Assume that there exist vertices \( u \) and \( w \) in \( G \) such that the vertex \( v \) lies on every \( u-w \) path of \( G \).

Then there are no \( u-w \) paths in \( G-v \), implying that \( G-v \) is disconnected and that \( v \) is a cut-vertex of \( G \).

**Examples:**

- The complete graphs have no cut-vertices
- Each nontrivial path contains only two vertices that are not cut-vertices

**Theorem 2.2:** Every nontrivial connected graph contains at least two vertices that are not cut-vertices.

**Proof.**

Let us suppose that the theorem false.

Then there exists a nontrivial connected graph containing at most one vertex that is not a cut-vertex. This means that every vertex of \( G \), with at most one exception, is a cut-vertex.

Let \( u \) and \( v \) be vertices of \( G \) such that \( d(u,v) = \text{diam } G \). At least one of \( u \) and \( v \) is a cut-vertex, say \( v \). Let \( w \) be a vertex belonging to a component of \( G-v \) not containing \( u \).

Since every \( u-w \) path in \( G \) contains \( v \), we conclude that \( d(u,w) > d(u,v) = \text{diam } G \),

which is a contradiction.
A bridge of a graph $G$ is an edge $e$ such that $k(G - e) > k(G)$.

- If $e$ is a bridge then $k(G - e) = k(G) + 1$.
- If $e = uv$ then $u$ is a cut-vertex of $G$ iff $\deg u > 1$.
- The complete graph $K_2$ is the only connected graph containing a bridge but no cut-vertex.

**Theorem 2.3.** An edge of a graph $G$ is a bridge iff there exist vertices $u$ and $w$ such that $e$ is on every $u$–$w$ path of $G$.

**Proof.**

Simple, homework.

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**Theorem 2.4.** An edge $e$ of a graph $G$ is a bridge of $G$ iff $e$ lies on no cycle of $G$.

**Proof.**

A: $\implies$

We can assume that $G$ is connected. Let $e = uv \in E(G)$. Suppose that $e$ lies on a cycle, we need to prove that $e$ is not a bridge.

Let $w_1$ and $w_2$ be arbitrary distinct vertices.

- If there exists always a $u$–$w_1$ path $P$ that $e$ does not lie on $P$, then $P$ is also a $w_1$–$w_2$ path of $G$–$e$, so $e$ is not a bridge.
- If $e$ lies on a $w_1$–$w_2$ path $Q$ of $G$, then replacing $e$ by an $u$–$v$ path on $C$ not containing $e$ produces a $w_1$–$w_2$ walk in $G$–$e$.

By Theorem 1.7 there is a $w_1$–$w_2$ path in $G$–$e$.

Thus, $w_1$ and $w_2$ are connected in $G$–$e$ and so $e$ is not a bridge.

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**B: $\implies$** Suppose that $e = uv$ is not a bridge of $G$. Then $G$–$e$ is connected.

Hence there exists a $u$–$v$ path $P$ in $G$–$e$.

But $P$ together with $e$ produces a cycle in $G$ containing $e$. 

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A cycle edge is an edge that lies on a cycle.

A bridge incident with an end-vertex is called a pendant edge.

A nontrivial connected graph with no cut-vertices is called a nonseparable graph.

A maximal nonseparable subgraph of $G$ is called a block.

If a connected graph contains a single block, then it is nonseparable.

Every two blocks have at most one vertex in common, namely a cut-vertex. (See the next slide.)
Theorem 2.5: A graph $G$ of order at least 3 is nonseparable iff every two vertices of $G$ lie on a common cycle of $G$.

Proof.

A: Let $G$ be a graph such that each two of its vertices lie on a cycle. Then $G$ is connected.

Suppose that $G$ is separable. Then $G$ has a cut-vertex, say $v$.

Then, by the Theorem 2.1, there exist vertices $u$ and $w$ such that $v$ is on every $u$-$w$ path in $G$.

Let $C$ be a cycle containing $u$ and $w$. This cycle determines two distinct $u$-$w$ paths, one of which does not contain $v$.

This contradict the fact that every $u$-$w$ path contains $v$. Therefore $G$ is nonseparable.

B: Let now $G$ be a nonseparable graph with at least three vertices. Let $u \in V(G)$ arbitrary, and we denote by $U$ all of the vertices that lie on a cycle containing $u$.

We will show that $U=V(G)$.

Assume $U \neq V(G)$. Then there exists a vertex $v \in V-U$.

Since $G$ is nonseparable, it contains no cut-vertices, and, since $|V(G)| \geq 3$, the graph does not contain bridge.

By Theorem 2.4, every edge lies on a cycle, and therefore if $w \in I(u)$ then $w \in U$.

Since $G$ is connected, there exists a $u$-$v$ path $u=u_0u_1\ldots u_k=v$ in $G$.

Let $i$ be the smallest integer, $2 \leq i \leq k$, such that $u_i \notin U$.

So, $u_{i-1} \in U$. 

C = $P_1 \cup P_2$
Let C be a cycle containing u and uᵢ. Since uᵢ is not a cut-vertex there exists a uᵢ-u path P: uᵢ = v₀, v₁,..., vₙ = u, so that uᵢ ∈ U.

There may be two distinct cases:

- The only vertex common to P and C is uᵢ, then there exists a cycle containing u and uᵢ, and this is a contradiction.
- P and C have a common vertex different from uᵢ, then let j be the smallest integer, 1 ≤ j ≤ l, such that vⱼ belongs to both P and C.

Now we can construct a cycle containing u and uᵢ in the following way:

- We start with the uᵢ-vⱼ subpath of P,
- We proceed along C from vⱼ to u and then uᵢ,
- We take the edge uᵢ,uᵢ.

This is again a contradiction.

An internal vertex of a u-v path P is any vertex of P different from u or v.

A set of paths, \{P₁, P₂, ..., Pₖ\}, is called internally disjoint if each internal vertex of Pᵢ, 1 ≤ i ≤ k, lies on no path Pⱼ, i ≠ j.

In particular, two u-v paths are internally disjoint if they have no vertices in common, other than u and v.

Similarly, edge-disjoint paths have no edges in common.

Corollary 2.6.: A graph G of order at least 3 is nonseparable iff there exist two internally disjoint u-v paths for every two distinct vertices u and v of G.

Theorem 2.7. (Harary and Norman, 1953): The center of every connected graph G lies in a single block of G.

Proof.

Suppose the opposite: there is a connected graph G whose center \text{Cent}(G) does not lie within a single block of G.

Then G has a cut-vertex v such that G-v contains components G₁ and G₂, each of which contains distinct vertices of \text{Cent}(G).

Let u be a vertex such that d(u,v)=e(v), and let P₁ be a v–u geodesic.

Let P₁ be a v–u geodesic.

At least one of G₁, G₂, say G₁, does not contain vertex from P₁.

Let w be a vertex of \text{Cent}(G) belonging to G₁, and let P₂ be a w–v geodesic.

The paths P₁ and P₂ form together a u–w path Pₙ, which is a u–w path of length d(u,w).

Then e(w) > e(v) which contradicts the fact that w is a central vertex.
2.3. The Reconstruction Problem (P.J. Kelly and S.M. Ulam, 1941)

A graph $G$ with $V(G) = \{v_i, v_2, \ldots, v_n\}$, $n \geq 2$, is said to be reconstructible if for every graph $H$ having $V(H) = \{u_1, u_2, \ldots, u_n\}$, $G - v_i = H - u_i$, for $i = 1, 2, \ldots, n$ implies that $G = H$.

If $G$ is reconstructible graph, then the subgraphs $G - v$, $v \in V(G)$, determine $G$ uniquely.

The Reconstruction Conjecture: Every graph of order at least 3 is reconstructible.

The Reconstruction Problem is thus to determine the truth or falsity of the Reconstruction Conjecture.

Exercises. (G. Chartrand and L. Lesniak page 38.)

1. Prove that if $v$ is a cut-vertex of a connected graph, then $v$ is not a cut-vertex of $\overline{G}$.

2. Prove the Theorem 2.3.

3. Prove that every graph containing only even vertices is bridgeless.

4. Let $G$ and $H$ be graphs with $V(G) = \{v_1, \ldots, v_n\}$ and $V(H) = \{u_1, \ldots, u_n\}$, $n \geq 3$.
   * Vertices $u_i$ and $u_j$ are adjacent in $H$ if $v_i$ and $v_j$ belong to a common cycle in $G$. Characterize those graphs $G$ for which $H$ is complete.
   * Vertices $u_i$ and $u_j$ are adjacent in $H$ if $deg v_i + deg v_j$ is odd in $G$. Prove that $H$ is bipartite.

Why we use the condition „order at least 3“?

Consider the graphs $G_1 = K_2$ and $G_2 = 2K_1$:

The subgraphs $G_1 - v$ where $v \in V(G_1)$ and the subgraphs $G_2 - v$, for $v \in V(G_2)$, are precisely the same:

$G_1 - v_1 \quad G_2 - v_1$

So, $G_1$ is not uniquely determined by its subgraphs, and the same is true for $G_2$.

The Reconstruction Conjecture claims that $K_2$ and $2K_1$ are the only nonreconstructible graphs.
Related problem: Given graphs $G_1, G_2, \ldots, G_n$ does there exist a graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $G_i = G - v_i$ for $i = 1, 2, \ldots, n$?

The answer is not known in general.

*Theorem 2.8.* (McKay 1977, and Nijenhuis 1977): If there is a counterexample to the Reconstruction Conjecture, then it must have order at least 12.

Proof. With the aid of computers they have shown that all graphs of order less than 12 are reconstructible.

There exists a graph $H$ with $V(H) = \{v_1, v_2, \ldots, v_6\}$ such that $G - v_i = H - v_i$ for $1 \leq i \leq 5$, but $G - v_6 \neq H - v_6$. Therefore the graphs $G - v_i$, $1 \leq i \leq 5$, do not determine $G$.

The graphs $G - v_i$, $4 \leq i \leq 6$, do uniquely determine $G$.

Digraphs are not reconstructible.

*Theorem 2.9.* (Stockmeyer, 1977): There are infinitely many pairs of counterexamples for digraph.
Theorem 2.10.: If $G$ is a graph of order $n \geq 3$ and size $m$, then $n$ and $m$ as well as the degree of the vertices of $G$ are determined from the $n$ subgraphs $G-v$, $v \in V(G)$.

Proof.

- To determine the number $n$ is trivial: it is one greater than the order of any subgraph $G-v$, $v \in V(G)$.

- To determine $m$, label these subgraphs by $G_i$, $i=1,2,\ldots,n$.

Let $V(G) = \{v_1,v_2,\ldots,v_n\}$ and suppose that $G_i = G - v_i$, where $v_i \in V(G)$. Let $m_i$ denote the size of $G_i$.

Consider an arbitrary edge $e$ of $G$, say $e=v_jv_k$.

Then $e$ belongs to $n-2$ of the subgraphs $G_i$, namely all except $G_j$ and $G_k$.

Hence $\sum_{i=1}^{n} m_i$ counts each edge $n-2$ times, that is $\sum_{i=1}^{n} m_i = (n-2)m$.

Therefore $m = \sum_{i=1}^{n} m_i$.

- The degrees of vertices of $G$ can be determined by simply noting that $\deg v_i = m - m_i$, $i=1,2,\ldots,n$.

A graphical parameter or a graphical property is recognizable if, for each graph $G$ of order at least 3, it is possible to determine the value of the parameter for $G$ or whether $G$ has the property from the subgraphs $G-v$, $v \in V(G)$.

Theorem 2.10. states that for a graph of order at least 3, the order, the size, and the degrees of its vertices are recognizable parameters.

As a consequence of the theorem it also follows that the property of graph regularity is recognizable.
**Theorem 2.11.** Every regular graph of order at least 3 is reconstructible.

Proof.

From the Theorem 2.10, it follows that the regularity and the degree of regularity are recognizable.

So, w.l.o.g. we may assume that $G$ is an $r$-regular graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$.

It remains to prove that $G$ is uniquely determined by its subgraphs $G-v_i$, $i=1,2,\ldots,n$.

Let $i$ be arbitrary, where $1 \leq i \leq n$, and consider the $G-v_i$ subgraph. Add a vertex $v_j$ to $G-v_i$. Since for any $v \in G-v_i$ the degree $\deg_{G-v_i}(v) = r-1$, we get the desired result.

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**Theorem 2.12.** If $G$ is a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$, then $G$ is connected if and only if at least two of the subgraphs $G-v_i$, $1 \leq i \leq n$, is connected.

Proof.

A: $\Rightarrow$

Let $G$ be connected. By Theorem 2.2, $G$ contains at least two vertices that are not-cut-vertices, implying the result.

B: $\Leftarrow$

Assume that there exist vertices $v_1, v_2 \in V(G)$ such that both $G-v_1$ and $G-v_2$ are connected.

Thus, in $G-v_1$, and also in $G$, $v_i$ is connected to each $v_j$, $i \geq 3$.

Hence every pair of vertices of $G$ are connected, and so $G$ is connected.

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**Theorem 2.13.** (proof by Manvel, 1970): Disconnected graph of order at least 3 are reconstructible.

Proof.

Since the property is recognizable so, w.l.o.g. we may assume that $G$ is a disconnected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$.

Let $G_i = G-v_i$ for $i=1,2,\ldots,n$. From Theorem 2.10, the degrees of the vertices $v_i$, $i=1,2,\ldots,n$, can be determined from the subgraphs $G_i$.

Hence, if $G$ contains an isolated vertex, then $G$ is reconstructible. So, we can assume that $G$ has no isolated vertices.

Since every component of $G$ is nontrivial, it follows that $k(G_j) \geq k(G)$ for $i=1,2,\ldots,n$ and that $k(G_j) = k(G)$ for some integer $j$ satisfying $1 \leq j \leq n$.

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So, the number of components of $G$ is $\min\{k(G_j) \mid j=1,2,\ldots,n\}$.

Suppose now, that $F$ is a component of $G$ of maximum order. Then $F$ must be a component of maximum order among the components of the graphs $G_i$. So, $F$ is recognizable.

Delete such a vertex from $F$ which is not a cut-vertex, getting $F'$.

Assume that there are $r$ (at least 1) components of $G$ isomorphic to $F$. We will see that $r$ is recognizable. Let

$$S = \{G_i \mid k(G_i) = k(G)\},$$

and let $S'$ be the subset of $S$ consisting of all those graphs $G_i$ having a minimum number $l$ of components isomorphic to $F$. (If $r=1$ then there exist graphs $G_j$ in $S$ containing no components isomorphic to $F$, that is $l=0$.)

Generally, $r = l + 1$. 

Graph Theory 2
Let now $S''$ denote the set of those graphs $G_i$ in $S'$ having a maximum number of components isomorphic to $F'$.

Assume that $G_1, G_2, \ldots, G_{t}$ ($t \geq 1$) are the elements of $S''$. Each graph $G_i$ in $S''$ has $k(G)$ components.

Since each graph $G_i$ ($1 \leq i \leq t$) has a minimum number of components isomorphic to $F$, each vertex $v_i$ ($1 \leq i \leq t$) belongs to a component $F_i$ of $G$ isomorphic to $F$, where the components $F_i$ of $G$ are not necessarily distinct.

Since each graph $G_i$ ($1 \leq i \leq t$) has a maximum number of components isomorphic to $F'$, it follows that $F_i - v_i = F'$ for each $i = 1, 2, \ldots, t$.

Hence, every two of the graphs $G_i, G_j, \ldots, G_t$ are isomorphic, and $G$ can be produced from $G_i$, say, by replacing a component of $G_i$ isomorphic to $F'$ by a component isomorphic to $F$.

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**Exercises.** (G. Chartrand and L. Lesniak page 52-53.)

1. Reconstruct the graph $G$ whose subgraphs $G - v, v \in V(G)$ are given in the following way:

   - $G_1$:
   - $G_2 = G_1$:
   - $G_3$:

2. Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_7\}$ such that $G - v_i = K_{2,2}$ for $i = 1, 2, 3$ and $G - v_i = K_{3,3}$ for $i = 4, 5, 6, 7$. Show that $G$ is reconstructible.

3. Prove that bipartiteness is a recognizable property.

4. Reconstruct the graph $G$ whose subgraphs $G - v, v \in V(G)$ are given in the following way:

   - $G_i$ ($i = 1, 2, \ldots, 8$):
   - $G_9 = G_{10}$:
   - $G_8$:

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$k(G) = \min\{ k(G_i) \mid i = 1, 2, \ldots, n \}$.
3. Trees and Labeling

- Simple theorem on trees
- Spanning trees
- Algorithms for finding a spanning tree
- Special trees: lines, double stars, caterpillars
- Labeling a graph: Cayley’s theorem on labeled trees
- Prüfer’s algorithm for labeling a graph
- Degree matrix and the Kirchoff Matrix-Tree Theorem
- Arboricity and vertex-arboricity

Among the connected graphs, the simplest yet most important are the trees.

A tree is an acyclic connected graph, a forest is an acyclic graph.

Elementary properties:
- every edge of a tree $T$ is a bridge.
- every block of a tree $T$ is acyclic.
- if $\deg v \geq 1$ then $T \setminus v$ is forest with $\deg v$ components.
- every vertex of $T$ that is not an end-vertex belongs to at least two blocks and is necessarily a cut-vertex.

Theorem 3.1.: Every nontrivial tree has at least two vertices of degree 1 (end-vertices).

Proof.
Let $d_1 \leq d_2 \leq \ldots \leq d_n$ be the degree sequence of a tree $T$ of order $n \geq 2$. Since $T$ is connected, so $\delta(T) = d_1 \geq 1$.

If $T$ had at most one vertex of degree 1, by the “Handshaking Lemma”, we would have

$$2m = 2(n-1) = 2n - 2 = \sum_{i=1}^{n} d_i \geq 1 + 2(n-1) = 2n - 1$$

This would be a contradiction.
Theorem 3.2: A graph $G(n,m)$ is a tree iff $G$ is acyclic and $n = m+1$.

Proof.

A: Assume that $G$ is a tree. Then $G$ is acyclic. We show that $m = n-1$ by induction on $n$.

1. For $n=1$, the result (and the graph) is trivial.

2. Assume that the equality $m = n-1$ holds for all trees with $n \geq 1$ vertices and $m$ edges, and let $T$ be a tree with $n+1$ vertices.

Let $v$ be an end-vertex of $T$.

The graph $T_v$ is a tree of order $n$ and so $T$ has $m = n - 1$ by the induction hypothesis.

Since $T$ has one more edge then $T_v$, it follows that $T$ has $m+1 = n$ edges.

Since $n+1 = (m+1) + 1$, the desired result follows.

B: Conversely, let $G(n,m)$ be an acyclic graph, where $m=n-1$. We need only verify that $G$ is connected.

Denote by $G_1, G_2, \ldots, G_k$ the components of $G$, where $G_i = G(n_i, m_i)$ ($1 \leq i \leq k$).

Since each $G_i$ is a tree, so $m_i = n_i - 1$. Hence,

$$n - 1 = m = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} (n_i - 1) = n - k$$

so that $k=1$ and $G$ is connected.
A well known problem in optimization theory asks for a relatively easy way of finding a spanning subgraph with a special property:

Given a graph $G=(V,E)$ and a positive valued cost function $f$ defined on the edges, $f:E\rightarrow \mathbb{R}^+$, find a connected spanning subgraph $T=\langle V,E \rangle$ of $G$ for which

$$f(T) = \sum_{xy \in E} f(xy)$$

is minimal.

We call such a spanning subgraph $T$ an economical spanning subgraph.

Example:

Since the system has to be economical, $T$ is a minimal connected spanning subgraph, that is a spanning tree of $G$.

Algorithm 1 (Kruskal, 1956): We choose one of the cheapest edges of $G$, that is an edge $e$ for which $f(e)$ is minimal. Each subsequent edge will be chosen from among the cheapest remaining edges of $G$ with the only restriction that we must not select all edges of any cycle.
Algorithm 2. This method based on the fact that it is foolish to use a costly edge unless it is needed to ensure the connectedness of the subgraph. So, let us delete one by one those costliest edges whose deletion does not disconnect the graph.

Algorithm 3. (Prim): Pick a vertex \( v_i \) of \( G \) and select one of the least costly edges incident with \( v_i \), say \( v_i v_j \). Then choose one of the least costly edges of the form \( v_i v_k \), where \( 1 \leq i \leq 2 \), and \( v_k \not\in \{v_i, v_j\} \). Having found vertices \( v_i, v_j, \ldots, v_k \) and an edge \( v_i v_j \), \( i < j \), for each vertex \( v_i \), with \( j \leq k \), select one of the least costly edges of form \( v_i v_k \), say \( v_i v_{k'} \), where \( 1 \leq i \leq k \) and \( v_k' \in \{v_1, v_2, \ldots, v_k\} \). The process terminates after we have selected \( n-1 \) edges.

Algorithm 4. (Boruvka): This method is applicable only if no two edges have the same cost. We start to choose the cheapest edge in each vertex simultaneously. At the end of this stage we get a spanning forest with components \( T_i \). Now, for each \( T_i \), we will choose a cheapest edge from those of ones where \( e = (x, y) \) and only \( x \in T_i \). We continue this procedure until the chosen edges form a connected graph.

Considering the Kruskal algorithm we can show the next: “philosophy”: In each step we choose the cheapest edge from the non-considered edges.

In other words, in each step we make the local best decision.

That algorithm which makes a local best solution in each step we call greedy algorithm.

We call a system of sets \( \mathcal{M} \) as a matroid if it has the following characteristics:

- \( \mathcal{M} \neq \emptyset \).
- If \( H \subseteq \mathcal{M} \) and \( I \supseteq H \) then \( I \in \mathcal{M} \).
- If \( A, B \in \mathcal{M} \) and \( |A| > |B| \) then there exists an \( a \in A \setminus B \) such that \( B \cup \{a\} \in \mathcal{M} \).
Theorem 3.7: Each of the four methods described above produces an economical spanning tree.

Proof. We will only prove that the Algorithm 1 produces an economical spanning tree. Choose an economical spanning tree $T$ of $G$ that has as many edges in common with $T_1$ as possible. Suppose that $E(T) \neq E(T_1)$. The edges of $T_i$ have been selected one by one. Let $xy$ be the first edge of $T_j$ that is not an edge of $T$. Then $T$ contains a unique $x-y$ path, say $P$. $T'$ contains more edges in common with $T_j$ as $T$ and it is also an economical spanning tree, contradicting the choice of $T$.

\[
\begin{align*}
G & \quad \text{uv} \not\in T_j \\
f(x, y) \leq f(u, v) & \quad T = T \cup xy \\
f(T') \leq f(T)
\end{align*}
\]

Theorem 3.8: A graph $G(n,m)$ is a tree iff $G$ is connected and $m = n-1$.

Proof.

A: Consider a tree $T(n,m)$. By definition, $T$ is connected and by the theorem 3.1, $m = n-1$.

B: Consider a $G(n,n-1)$ connected graph. We need to show that $G$ is acyclic.

Let us suppose the opposite, i.e. $G$ has a cycle $C$. Let $e$ be an edge of $C$.

Then $C-e$ is a connected graph that $C(n,n-2)$. It is impossible.

Theorem 3.9: A sequence $d_1,d_2,...,d_n$ of $n \geq 2$ positive integers is the degree sequence of a tree of order $n$ iff

\[
\sum_{i=1}^{n} d_i = 2n - 2.
\]

Proof.

A: Let us consider a tree $T(n,m)$ and a degree sequence $d_1,d_2,...,d_n$. Then

\[
\sum_{i=1}^{n} d_i = 2m = 2(n-1) = 2n - 2.
\]

B: By induction.

- If $n = 2$ then the only sequence of two positive integers with sum equal to 2 is 1,1, and this is the degree sequence of the tree $K_2$.
- Assume that whenever a sequence of $n-1 \geq 2$ positive integers has the sum $2(n-2) = 2n-4$, then it is the degree sequence of a tree of order $n-1$.

Let $d_1,d_2,...,d_n$ be a sequence of $n$ positive integers with

\[
\sum_{i=1}^{n} d_i = 2n - 2
\]

and suppose that $d_1 \geq d_2 \geq ... \geq d_n$. We show that this is a degree sequence of a tree.

Since each term $d_i$ is a positive integer, it follows that $2 \leq d_i \leq n-1$ and $d_{n+1} = d_n = 1$. 


Hence $d_1, d_2, d_3, \ldots, d_{n-1}$ is a sequence of $n-1$ positive integers whose sum is $2n-4$.

By the induction hypothesis, there exists a tree $T'$ of order $n-1$ with

$$V(T') = \{v_1, v_2, \ldots, v_{n-1}\}$$

such that $\deg v_i = d_i$ for $2 \leq i \leq n-1$.

Let $T$ be the tree obtained from $T'$ by adding a new vertex $v_n$ and joining it to $v_1$.

Thus $d_1, d_2, \ldots, d_n$ is a degree sequence of $T$.

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A tree $T$ is a double star if it contains exactly two vertices that are not end-vertices. (Necessarily, these vertices are adjacent.)

A caterpillar is a tree $T$ with the property that the removal of the end-vertices of $T$ results in a path. This path is referred to as the spine of the caterpillar.

- If the spine is trivial then the caterpillar is a star
- If the spine is $K_2$, then the caterpillar is a double star.

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Theorem 3.10: Let $T$ be a nontrivial tree with $A(T) = k$, and let $n_i$ be the number of vertices of degree $i$ for $i=1, 2, \ldots, k$. Then

$$n_i = n_i + 2n_{i+1} + 3n_{i+2} + \ldots + (k-2)n_{k-2} + 2.$$  

**Proof.**

Suppose that $T$ has order $n$ and size $m$. Then

$$\sum_{i=1}^{k} n_i = n$$

and

$$\sum_{i=1}^{k} in_i = 2m = 2n - 2 = 2\sum_{i=1}^{k} n_i - 2$$

or

$$\sum_{i=1}^{k} (i-2)n_i + 2 = 0. \quad (1)$$

Solving (1) for $n_i$ gives the desired result.
Theorem 3.11.: Let \( T \) be a tree of order \( k \), and let \( G \) be a graph with \( \delta(G) \geq k-1 \). Then \( G \) contains a subgraph isomorphic to \( T \).

Proof. By induction on \( k \).

- The result is obvious for \( k=1 \) since \( K_1 \) is a subgraph of every graph. Similarly, the statement is true for \( k=2 \) since \( K_2 \) is a subgraph of every nonempty graph.
- Assume for each tree \( T' \) of order \( k-1, k \geq 3 \), and every graph \( H \) with \( \delta(H) \geq k-2 \), that contains a subgraph isomorphic to \( T' \).

Let \( T \) be a tree of order \( k \) and let \( G \) be a graph with \( \delta(G) \geq k-1 \). We show that \( G \) contains a subgraph isomorphic to \( T \).

Let \( v \) be an end-vertex of \( T \) and let \( u \) be the vertex of \( T \) adjacent to \( v \). The graph \( T-v \) is necessarily a tree of order \( k-1 \).

The graph \( G \) has \( \delta(G) \geq k-1 > k-2 \), so by the induction hypothesis, \( G \) contains a subgraph \( F \) isomorphic to \( T-v \).

Let \( u' \) denote the vertex of \( F \) that corresponds to \( u \). Since \( \deg_G u' \geq k-1 \) and \( T-v \) has order \( k-1 \), the vertex \( u' \) is adjacent to a vertex of \( G \) that does not belong to \( F \).

Therefore, \( G \) contains a subgraph isomorphic to \( T \).

3.2. Labeling a graph.

If we sign to all to vertices of \( G(n, m) \) a label from a given label set of \( n \) elements, then we speak about a labeled graph.

Two labelings of a graph \( G \) of order \( n \) from the same set of \( n \) labels are considered distinct if they do not produce the same edge set.
Theorem 3.12. (Cayley, 1889): There are $n^{n-2}$ distinct labeled trees of order $n$.

Proof.

The proof is due to Prüfer (1914). He proved the statement of the theorem by construction:

Suppose there is given a tree $T$ which vertices are labelled from the set $\{1, 2, ..., n\}$.

We will construct a sequence of length $n-2$ (Prüfer sequence), and we state a one-to-one correspondence between a labelled graph and the corresponding Prüfer sequence.

Since the number of Prüfer sequences is $n^{n-2}$ the construction proves the theorem.
Fact 1: Each choice is unique so the correspondence is also unique.

Fact 2: Every vertex \( v \) of \( T \) appears in its Prüfer sequence \( \deg v - 1 \) times.

Fact 3: No end-vertex of \( T \) appear in the Prüfer sequence for \( T \).

Fact 4: If \( T \) is a tree of order \( n \) and size \( m \), then the number of terms in its Prüfer sequence is

\[
\sum_{v \in V(T)} (\deg v - 1) = 2m - n = 2(n - 1) - n = n - 2
\]

Now we consider the converse question, that is, if \( (p_1, p_2, \ldots, p_{n-2}) \) is a sequence of length \( n-2 \) such that each \( p_i \in \{1, 2, \ldots, n\} \), then we construct a unique labeled tree \( T \) of order \( n \) such that the given sequence is the Prüfer sequence for \( T \).
Not that \( n^2 \) is not the number of nonisomorphic spanning trees of \( K_n \), but the number of distinct (labeled) spanning trees of \( K_n \). There are just six nonisomorphic spanning trees of \( K_6 \), whereas there are 64 = 1296 distinct spanning trees of \( K_6 \).

The degree matrix \( D(G) = [d_{ij}] \) in the \( n \times n \) matrix with \( d_{ii} = \text{deg} v_i \) and \( d_{ij} = 0 \) for \( i \neq j \).

**Theorem 3.13.** (Kirchhoff, 1847) Matrix-Tree Theorem: If \( G \) is a nontrivial labeled graph with adjacency matrix \( A \) and degree matrix \( D \), then the number of distinct spanning trees of \( G \) is the value of any cofactor of the matrix \( D - A \).

We do not prove the theorem but illustrate with an easy example:

\[
D = \begin{bmatrix}
2 & -1 & -1 & 0 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 3 & -1 \\
0 & -1 & -1 & 2
\end{bmatrix}
\]

\[
D - A = \begin{bmatrix}
2 & -1 & -1 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]

Now we count the cofactor of the matrix \( D - A \):

\[
\begin{vmatrix}
2 & -1 & -1 & 0 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 3 & -1 \\
0 & -1 & -1 & 2
\end{vmatrix} = (-1)^{2+3} \begin{vmatrix}
2 & 0 \\
0 & -1 & 2
\end{vmatrix} = 8
\]

Consequently, there are 8 distinct spanning trees of the given graph \( G \). We show them on the next slide.
Exercises. (G. Chartrand and L. Lesniak, page 65.)

1. Determine the Prüfer sequence of trees on slide 20.

2. Determine the labeled tree having Prüfer sequence (4, 5, 7, 2, 1, 6, 6, 7).

3. Let $G$ be the labeled graph below:

```
  v1  v2  v3  v4  v5
   |   |   |   |
  +---+---+---+---+
   |   |   |   |   |
  v6  v7  v8  v9
```

- Use the Matrix-Tree Theorem to compute the number of distinct labeled spanning trees of $G$.
- Draw all the distinct labeled spanning trees of $G$.

Arboricity and vertex-arboricity

One of the most common problems in graph theory deals with decomposition of a graph into various subgraphs possessing some prescribed property. There are two problems of this type:

- decomposition of the vertex set
- decomposition of the edge set.

One such property that has been the subject of investigation is that of being acyclic.

Fact: For a graph $G$ it is always possible to partition $V(G)$ into subsets $V_i$, $1 \leq i \leq k$, such that each induced subgraph $G[V_i]$ is acyclic.

Example: we select each subset $V_i$ so that $|V_i| \leq 2$.

The major problem is to partition $V(G)$ so that as few subsets as possible are involved:

The vertex-arboricity $a(G)$ of a graph $G$ is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces an acyclic subgraph.

Some easy examples:

- $a(G) = 1$ if $G$ is acyclic.
- $a(C_n) = 2$.
- $a(K_n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \lfloor n+1 \rfloor/2 & \text{if } n \text{ is odd} \end{cases}$

No formula is known in general, but some bounds for this number exist.

Fact: For every graph $G$ of order $n$ $a(G) \leq \lceil n/2 \rceil$.

A graph $G$ is called critical with respect to vertex-arboricity if $a(G-v) < a(G)$ for all vertices $v$ of $G$. The graph $G$ is k-critical if $a(G) = k$.

Theorem 3.14.: Every graph $G$ with $a(G) = k \geq 2$ contains an induced $k$-critical subgraph.

Proof.

The statement follows immediately from the fact that for every induced subgraph $G'$ of $G$ of minimum order with $a(G') = k$ is $k$-critical.
Theorem 3.15.: If G is a graph having \( a(G) = k \geq 2 \) that is critical with respect to vertex-arboricity, then \( \delta(G) \geq 2k-2 \).

Proof.

Let \( G \) be a \( k \)-critical graph, \( k \geq 2 \), and suppose that \( G \) contains a vertex \( v \) with \( \text{deg} v \leq 2k-3 \).

Since \( G \) is \( k \)-critical, \( a(G-v) = k-1 \) and there is a partition \( V_1, V_2, \ldots, V_{k-1} \) of the vertex set of \( G-v \) such that each subgraphs \( (V_i) \) is acyclic.

Since \( \text{deg} v \leq 2k-3 \), at least one of the subsets, say \( V_r \), contains at most one vertex adjacent with \( v \) in \( G \).

The subgraph \( (V_r \cup \{v\}) \) is necessarily acyclic.

Hence \( V_1, V_2, \ldots, V_r \cup \{v\}, \ldots, V_{k-1} \) is a partition of the vertex set of \( G \) into \( k-1 \) subsets, each of which induces an acyclic subgraph.

This contradicts the fact that \( a(G) = k \).

Theorem 3.16.: For each graph \( G \),

\[
a(G) \leq 1 + \left\lfloor \frac{\delta(G')}{2} \right\rfloor,
\]

where the maximum is taken over all induced subgraphs \( G' \) of \( G \).

Proof.

The result is obvious for acyclic graphs. Thus let \( G \) be a graph with \( a(G) = k \geq 2 \).

Let \( H \) be an induced \( k \)-critical subgraph of \( G \). Since \( H \) itself is an induced subgraph of \( G \), so

\[
\delta(H) \leq \max \delta(G'),
\]

where the maximum is taken over all induced subgraphs \( G' \) of \( G \). By the previous theorem \( \delta(H) \geq 2k-2 \), so

\[
\max \delta(G') \geq 2k - 2 = 2a(G) - 2.
\]

This inequality produces the desired result.

Corollary 3.17.: For every graph \( G \),

\[
a(G) \leq 1 + \left\lfloor \frac{\Delta(G)}{2} \right\rfloor,
\]

where the maximum is taken over all induced subgraphs \( G' \) of \( G \).

Proof.

The statement follows immediately from the fact that \( \delta(G') \leq \delta(G) \).

The edge-arboricity \( a_e(G) \) of a nonempty graph \( G \) is the minimum number of subsets into which \( E(G) \) can be partitioned so that each subset induces an acyclic subgraph.

Theorem 3.18.: For each graph \( G \),

\[
a_e(G) \geq 1 + \left\lfloor \frac{\max \delta(G')}{2} \right\rfloor,
\]

where the maximum is taken over all induced subgraphs \( G' \) of \( G \).

Proof.

Let \( G \) be an induced subgraph of \( G \) having order \( n_i \) and size \( m_i \).

Thus \( G_i \) can be decomposed into \( a_e(G) \) or fewer acyclic subgraphs, each of which has size at most \( n_i - 1 \). Then

\[
\delta(G_i) \leq \frac{2m_i}{n_i} \leq 2(n_i - 1) \frac{a_e(G)}{n_i} < 2a_e(G).
\]

Hence

\[
\max \delta(G') < 2a_e(G),
\]

where the maximum is taken over all induced subgraphs \( G' \) of \( G \).

Therefore \( 2a_e(G) \geq 1 + \max \delta(G') \), which yields the desired result.
Corollary 3.19: For every graph $G$, $a(G) \leq a_1(G)$.

Proof.

By Theorems 3.11 and 3.12 we have

$$2a(G) - 2 \leq \max \delta(G') \leq 2a_1(G) - 1,$$

where the maximum is taken over all induced subgraphs $G'$ of $G$. So $a(G) \leq a_1(G) + \frac{1}{2}$.

Since $a(G)$ and $a_1(G)$ are integers, the result follows.