5. Hamiltonian Graphs

- The Traveling Salesman Problem.
- Hamiltonian paths and hamiltonian cycles.
- Characterization of hamiltonian graphs.
- The closure of a graph.
- Chvatal's theorem for hamiltonicity.
- Erdős-Chvatal Theorem.
- "Highly" hamiltonian and "nearly" hamiltonian graphs.
- Hamiltonian decomposition of graphs.
- Hamiltonian digraphs.
- Line graphs and the powers of a graph.
- Dominating circuits.

5.1. Hamilton cycles and Hamilton paths

The traveling salesman problem: a salesman is to make a tour of \( n \) cities, at the end of which he has to return to the head office he starts from. The cost of the journey between any two cities is known. The problem asks for an efficient algorithm for finding a least expensive tour.

What is "efficient algorithm"?

It is not known whether or not there is an efficient algorithm for finding a least expensive route.

Variant: the route is required to be a cycle, that is the salesman is not allowed to visit the same city twice.

A cycle containing all the vertices of a graph is said to be a Hamiltonian cycle.

A graph containing a Hamilton cycle is said to be Hamiltonian.

The origin of this term is a game invented in 1857 by Sir William Rowan Hamilton based on the construction of cycles containing all vertices in the graph of the dodecahedron. The "Rubik cube" of the 19th century.

In 1855 Thomas P. Kirkman posed the following question: Given the graph of a polyhedron, can one always find a circuit that passes through each vertex once and only once?
In fact, Hamilton cycles and paths in special graphs had been studied well before Hamilton proposed his game.

The puzzle of the knight's tour on a chessboard, thoroughly analysed by Euler in 1759, asks for a Hamilton cycle in the graph whose vertices are the 64 squares of a chessboard and in which two vertices are adjacent if a knight can jump from one square to the other.

**Theorem 5.2:** If \( G \) is a Hamiltonian graph, then for every \( S \neq \emptyset \) and \( S \subseteq V(G) \),

\[
k(G - S) \leq |S|.
\]

**Proof.**

Let \( S \) be a proper nonempty subset of \( V(G) \), and suppose that \( k(G - S) = k \geq 1 \), where \( G_1, G_2, \ldots, G_k \) are the components of \( G - S \). Let \( C \) be a Hamiltonian cycle of \( G \).

When \( C \) leaves \( G_j \) (\( 1 \leq j \leq k \)), then the next vertex of \( C \) belongs to \( S \).

Thus \( k(G - S) \leq |S| \).

If \( G \) is Hamiltonian, then \( G \) is connected and contains a Hamiltonian cycle.

**Theorem 5.1:** If \( G \) is Hamiltonian then \( G \) is 2-connected.

**Corollary 5.3:** Every Hamiltonian graph is 3-connected.
Theorem 5.4. (O. Ore, 1960): If \( G \) is a graph of order \( n \geq 3 \) such that for all distinct nonadjacent vertices \( u \) and \( v \)

\[
deg u + \deg v \geq n
\]

then \( G \) is hamiltonian.

Proof.

Suppose that the theorem is not true, i.e. there exists a maximal nonhamiltonian graph \( G \) of order \( n > 3 \) that satisfies the hypothesis of the theorem.

Then \( G \) is nonhamiltonian – so, it is not complete - and every two nonadjacent vertices \( w_1 \) and \( w_2 \) of \( G \), the graph \( G + w_1w_2 \) is hamiltonian.

Let \( u \) and \( v \) two nonadjacent vertices of \( G \). Thus, \( G + uv \) is hamiltonian and every hamiltonian cycle of \( G + uv \) contains the edge \( uv \).

Then there is a \( u - v \) path \( P: u = u_1, u_2, \ldots, u_n = v \) containing every vertex of \( G \).

If \( u_iu_j \in E(G), 2 \leq i \leq n \), then \( u_iu_j \notin E(G) \). For otherwise

\[
u_1, u_2, u_3, \ldots, u_i, u_{i+1}, u_{i+2}, \ldots, u_n
\]

is a hamiltonian cycle of \( G \).

Hence for each vertex of \( \{u_2, u_3, \ldots, u_n\} \) adjacent to \( u_i \) there exists at least one vertex from \( \{u_1, u_2, \ldots, u_{i-1}\} \) which is not adjacent with \( u_i \).

So

\[
\deg u_i \leq (n-1) - \deg u_i
\]

and this implies that there is a nonadjacent vertex-pair in \( G \) for which

\[
\deg u + \deg v \leq n - 1
\]

which is a contradiction.
**Theorem 5.5.** (J. A. Bondy, V. Chvatal, 1976): Let $u$ and $v$ be distinct nonadjacent vertices of a graph $G$ of order $n$ such that $\deg u + \deg v \geq n$. Then $G + uv$ is hamiltonian if $G$ is hamiltonian.

**Proof.**

A: $\Rightarrow$ If $G$ is hamiltonian and $u$ and $v$ are nonadjacent vertices, then $G + uv$ is also hamiltonian.

B: $\Leftarrow$ Suppose that $G$ is a graph of order $n$ with nonadjacent vertices $u$ and $v$ such that $\deg u + \deg v \geq n$ and $G + uv$ is hamiltonian.

If $G$ is not hamiltonian then – similarly to the proof of the Theorem 5.4. – we arrive at the contradiction $\deg u + \deg v \leq n - 1$.

So, $G$ must be hamiltonian.

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**Theorem 5.6.** If $G_1$ and $G_2$ are two graphs obtained from a graph $G$ of order $n$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $n$, then $G_1 = G_2$.

**Proof.**

Let $e_1, e_2, \ldots, e_r$ and $f_1, f_2, \ldots, f_s$ be the sequences of edges added to $G$ obtained $G_1$ and $G_2$, respectively.

It is enough to show that each $e_i (1 \leq i \leq r)$ is an edge of $G_2$ and that each $f_i (1 \leq i \leq s)$ is an edge of $G_1$.

Assume, to the contrary, that this is not the case, i.e. for some $t, 0 \leq t \leq r - 1$, $e_t \notin E(G_2)$ if $i \leq t$, and the edge $e_{i+1} = uv$ does not belong to $G_2$.

Let $G_j$ the graph obtained from $G$ by adding the edges $e_1, e_2, \ldots, e_r$.

It follows from the definition of $G_j$ that $\deg G_j(u) + \deg G_j(v) \geq n$.

This is a contradiction, since $u$ and $v$ are nonadjacent vertices of $G_2$ so while constructing $G_2$, we must choose it.

Thus each $e_i$ belongs to $G_2$, and – by the symmetricity – we can also prove that each $f_i$ belongs to $G_j$.

So, $G_j = G_j$.

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**Theorem 5.7.** A graph is hamiltonian iff its closure is hamiltonian.

**Proof.**

The Theorem is a simple consequence of the definition of closure and the Theorem 5.5.
**Theorem 5.8.** Let $G$ be a graph with at least three vertices. If $C(G)$ is complete, then $G$ is hamiltonian.

**Proof.**

The statement follows from the fact that each complete graph is hamiltonian.

**Theorem 5.9.** (V. Chvatal, 1972): Let $G$ be a graph of order $n \geq 3$, the degrees $d_i$ of whose vertices satisfy $d_1 \leq d_2 \leq \ldots \leq d_n$. If there is no integer $k < n/2$ for which $d_k \leq k$ and $d_k \leq n - k - 1$, then $G$ is hamiltonian.

**Proof.**

We will show that $C(G)$ is complete which, by the Theorem 5.8, implies that $G$ is hamiltonian.

Assume, to the contrary, that $C(G)$ is not complete.

Let $u$ and $w$ be nonadjacent vertices of $C(G)$ for which $\deg_{C(G)} u + \deg_{C(G)} w$ is as large as possible.

Since $u$ and $w$ are nonadjacent it follows that $\deg_{C(G)} u + \deg_{C(G)} w \leq n - 1$.

W.l.o.g. we can suppose that $\deg_{C(G)} u \leq \deg_{C(G)} w$.

If a graph $G$ satisfies the condition of Theorem 5.4, then $C(G)$ is complete and so, by Theorem 5.8, $G$ is hamiltonian.

Thus, Ore’s theorem is an immediate corollary of Theorem 5.8 (although chronologically it preceeded the theorem of Bondy and Chvatal by several years).

Perhaps surprisingly, many well-known sufficient condition for a graph to be hamiltonian based on vertex degrees can be deduced from Theorem 5.8.

Theorem 5.9. due to Chvatal, is an example of one of the strongest of these.
Suppose to obtain a cycle having length at least |V(C)|+1. Then the result follows since $C(G)$ is hamiltonian.

We can suppose: $\deg_{C(G)} u \leq \deg_{C(G)} w$.

\[ |W| = (n - 1) - \deg_{C(G)} w \geq k. \]

**Theorem 5.11** (P. Erdős, V. Chvatal, 1972): Let $G$ be a graph with at least three vertices. If $\kappa(G) \geq \beta(G)$ then $G$ is hamiltonian.

Proof.

If $\beta(G) = 1$ then the result follows since $G$ is complete.

Assume that $\beta(G) \geq 2$. Let $\kappa(G) = k$.

Since $k \geq 2$, $G$ contains at least one cycle. Among all cycles of $G$, let $C$ be a maximum length cycle.

By **Theorem 4.15**, there are at least $k$ vertices on $C$. We will show that that $C$ is hamiltonian.

Suppose the contrary. Then there exists a vertex $w$ of $G$ that does not lie on $C$.

**Corollary 5.10** (Dirac, 1952): If $G$ is a graph of order $n \geq 3$ such that $\deg v \geq n/2$ for every vertex of $G$, then $G$ is hamiltonian.

In the case of regular graphs this statement can be improved: Jackson showed in 1980 that every 2-connected $r$-regular graph of order at most $3r$ is hamiltonian. (The Petersen graph shows that $3r$ cannot be replaced by $3r+1$.)

**Graph Theory 5**

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Since $|V(C)| \geq k$, we may apply the **Theorem 4.14** to conclude that there are $k$ paths $P_1, P_2, \ldots, P_k$ having initial vertex $w$ that are pairwise disjoint, except for $w$, and that share with $C$ only their terminal vertices $v_1, v_2, \ldots, v_k$ respectively.

If any two of the vertices $v_i$ are consecutive on $C$, then there is a cycle containing more vertices than $C$ has.

For each $i = 1, 2, \ldots, k$, let $u_i$ be the vertex following $v_i$ in some fixed cyclic ordering of $C$.

No vertex $u_i$ is adjacent to $w$ in $G$; otherwise we could replace the edge $v_i, u_i$ in $C$ by the $v_i - u_i$ path determined by the path $P_i$ and the edge $u_i w$ to obtain a cycle having length at least $|V(C)|+1$, which is impossible.
A path in a graph $G$ containing every vertex of $G$ is called a hamiltonian path.

A graph is hamiltonian-connected if for every pair $u,v$ of distinct vertices, there exists a hamiltonian $u-v$ path.

A hamiltonian-connected graph with at least three vertices is hamiltonian.

The $(n+1)$-closure $C_n(G)$ of a graph $G$ of order $n$ to be the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $n+1$ until no such pair remains.

### Theorem 5.12 (J. A. Bondy and V. Chvatal, 1976)

Let $G$ be a graph of order $n$. If $C_n(G)$ is complete, then $G$ is hamiltonian-connected.

**Proof.**

If $n = 1$, then the result is obvious, so we can suppose that $n \geq 2$.

Let $G$ be a graph of order $n$ whose $(n+1)$-closure is complete, and let $u$ and $v$ be any two vertices of $G$.

Let $H$ be a graph which consist of $G$ together with a new vertex $w$ and the edges $uw$ and $vw$. So, $H$ has order $n + 1$.

Since $C_{n+1}(G)$ is complete, so $|V(G)|_{(n+1)} = K_n$.

Thus $\deg_{C(H)} x \geq n - 1$ for $x \in V(G)$. So

\[
\deg_{C(H)} x + \deg_{C(H)} w \geq n + 1.
\]

Therefore $C(H) = K_{n+1}$, and by Theorem 5.8, $H$ is hamiltonian.

Any hamiltonian cycle of $H$ contains the edges $uw$ and $vw$, so $G$ has a hamiltonian $u-v$ path.

### Corollary 5.13.

If $G$ is a graph of order $n$ such that for all distinct nonadjacent vertices $u$ and $v$, 

\[
\deg u + \deg v \geq n + 1,
\]

then $G$ is hamiltonian-connected.

### Corollary 5.14.

If $G$ is a graph of order $n$ such that \(\deg v \geq (n+1)/2\) for every vertex $v$ of $G$, then $G$ is hamiltonian-connected.

### Corollary 5.15.

Let $G$ be a graph of order $n \geq 3$, the degrees $d_i$ of whose vertices satisfy 

\[d_1 \leq d_2 \leq \ldots \leq d_n\]

If there is no integer $k < n/2$ for which $d_k \leq k$ and $d_{n-k} \leq n-k-1$, then $G$ is hamiltonian-connected.
Restrictive version of the Traveling Salesman Problem: there are only two travel costs, 1 and \( \infty \) (expressing the impossibility of the journey), then the question is whether or not the graph formed by the edges with travel cost 1 contains a Hamilton cycle.

**Is there efficient algorithm?**

Even this special case of the TSP is unsolved: no efficient algorithm is known for constructing a Hamilton cycle, though neither is it known that there is such algorithm.

If the travel cost between any two cities is the same, then our salesman has no difficulty in finding a least expensive tour: any permutation of the \( n-1 \) cities will do. (The \( n \)th city is the head office.)

**Rewelling in his new found freedom, our salesman decides to connect “duty and pleasure” and promises not to take the same road \( xy \) again while there is a road \( uv \) has not seen yet.**

**Can he keep his promise?**

To plan a required sequence of journeys we have to decompose \( K_n \) into disjoint Hamilton cycles.

**For which values of \( n \) is this possible?**

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**Theorem 5.16.** For \( n \geq 2 \) the complete graph \( K_n \) is decomposable into edge disjoint Hamiltonian paths if and only if \( n \) is even.

Let us denote the size of a graph \( G \) by \( e(G) \).

**A:**\( \iff \)

- \( e(K_n) = \frac{1}{2} n(n-1) \)
- A Hamilton path in a graph with \( n \) vertices has \( n-1 \) edges.

**B:**\( \iff \)

Let us assume now that \( n \) is even and \( n \geq 2 \). We will prove by construction:

- We will have \( n/2 \) Hamilton path.
- For each vertex \( v_i \in G \) \( \deg v_i = n - 1 \).
- In the construction a vertex will be exactly once either a start- or an endvertex in Hamilton paths.
Theorem 5.17.: For $n \geq 3$ the complete graph $K_n$ is decomposable into edge disjoint Hamiltonian cycles iff $n$ is odd.

Proof.

A: $\iff$ Let us suppose that $K_n$ is decomposable.

Since $K_n$ is $(n-1)$-regular and a cycle is 2-regular, then $n-1$ must be even.

\[ n \text{ is odd.} \]

The statement also follows immediately from the fact that size of $K_n$ is

\[ \frac{1}{2}n(n - 1) \]

B: $\leftarrow$ Let us assume now that $n$ is odd and $n \geq 3$.

- Let us leave a vertex.
- The remaining graph is $K_{n-1}$, where $n-1$ is even, so it is decomposable into the union of disjoint Hamilton paths.
- Let us join the endvertices of each path with the vertex we left before.
- So we have got Hamilton cycles, and they are disjoint.
5.2. Hamiltonian Digraphs.

A digraph $D$ is hamiltonian if it contains a spanning cycle; such a cycle is called hamiltonian cycle.

- The situation for hamiltonian digraphs is even more complex than it is for hamiltonian graphs.
- As with hamiltonian graphs, no characterization of hamiltonian digraphs exists.
- While there are sufficient conditions for a digraph to be hamiltonian, they are analogues of the simpler sufficient conditions for hamiltonian graphs.

Proof.

First we show that the condition in the statement implies that $D$ is strong.

Let $u$ and $v$ any two vertices of $D$. We will show that there is a $u$–$v$ path in $D$:

If $(u,v) \notin E(D)$ then the result is obvious.

If $(u,v) \notin E(D)$ then by the condition there must exist a vertex $w$ in $D$, with $w \neq u$, $v$ such that both, $(u,w)$ and $(w,v) \notin E(D)$.

However, then $u$, $w$, $v$ is the desired $u$–$v$ path, so $D$ is strong.

Let $u$ and $v$ be any two nonadjacent vertices of $D$. Then by the condition in the statement it follows that $\deg u + \deg v \geq 2n$.

So, $\deg u + \deg v \geq 2n$.

Thus, by the Theorem 5.18, $D$ is hamiltonian.

A digraph $D$ is strong if for every two distinct vertices $u$ and $v$ of $D$, there is both $u$–$v$ (directed) and $v$–$u$ path.

Theorem 5.18.: If $D$ is a nontrivial strong digraph of order $n$ such that $\deg u + \deg v \geq 2n - 1$ for every pair $u$, $v$ of distinct nonadjacent vertices, then $D$ is hamiltonian.

The proof is an immediate consequence of the Theorem 5.19.

Corollary 5.19.: If $D$ is a nontrivial strong digraph of order $n$ such that whenever $u$ and $v$ are distinct vertices and $(u,v) \notin E(D)$

$$\deg u + \deg v \geq n,$$

then $D$ is hamiltonian.

Corollary 5.20.: If $D$ is a strong digraph of order $n$ such that $\deg v \geq n$ for every vertex $v$ of $D$, then $D$ is hamiltonian.

Corollary 5.21.: If $D$ is a digraph of order $n$ such that

$$\deg v \geq n/2 \text{ and } \deg v \geq n/2$$

for every vertex $v$ of $D$, then $D$ is hamiltonian.
Exercises. (G. Chartrand and L. Lesniak page 104.)

1. Show that the graph of the dodecahedron is hamiltonian.

2. Give an example of a \( l \)-tough graph that is not hamiltonian.

3. Prove that if \( G \) and \( H \) are hamiltonian graphs then \( G \times H \) is hamiltonian.

4. Prove that the \( n \)-cube \( Q_n \), \( n \geq 2 \), is hamiltonian.

5.3. Line Graphs and Powers of Graphs

The \( k \)th power \( G^k \) of a connected graph \( G \), where \( k \geq 1 \), is that graph with \( V(G^k) = V(G) \) for which \( uv \in E(G^k) \) iff \( d_G(u, v) \leq k \).

- \( G^2 \) is a square of \( G \).
- \( G^3 \) is a cube of \( G \).

What can we say about the hamiltonicity of a power-graph?

\( G^3 \) is hamiltonian if \( G \) is hamiltonian.

Proof.

Since \( G^3 \) contains \( G \) as a subgraph, the statement follows immediately.

On the other side, for any connected graph \( G \) – independently whether it is hamiltonian or not – of order at least 3 and for a sufficient large \( k \), the graph \( G^k \) is hamiltonian since \( G^3 \) is complete if \( G \) has diameter \( d \).

What is the minimum \( k \) for which \( G^3 \) is hamiltonian?

The next slide shows an example where \( G^2 \) is not hamiltonian, but \( G^3 \) is hamiltonian.

\[ G: \quad G^2: \quad G^3: \]

A graph whose square is not hamiltonian.
Theorem 5.22 (Karaganis, 1968 Sekanina 1960): If $G$ is a connected graph, then $G^3$ is hamiltonian-connected.

Proof.

If $H$ is a spanning tree of $G$ and $H^3$ is hamiltonian-connected, then $G^3$ is hamiltonian-connected.

It is sufficient to prove that the cube of every tree is hamiltonian-connected.

We will show it by induction on $n$, the order of the tree.

- For small values of $n$ the result is obvious.
- Assume for all trees $H$ of order less than $n$ that $H^3$ is hamiltonian-connected, and let $T$ be a tree of order $n$, and let $u$ and $v$ be any two vertices of $T$.

Case 1: Suppose that $u$ and $v$ are adjacent in $T$.

Let $e = uv$, and consider the forest $T-e$. This forest has two components $T_u$ and $T_v$ containing $u$ and $v$, respectively.

By the induction hypothesis $T_u^3$ and $T_v^3$ are hamiltonian-connected.

Let $u_i$ be any vertex of $T_u$ adjacent to $u$, and let $v_j$ be any vertex of $T_v$ adjacent to $v$. (If $T_u$ or $T_v$ is trivial, then we define $u = u_i$ or $v = v_j$, respectively.)

Since $d_T(u_i, v_j) \leq 3$ so it is true that $u_i$ and $v_j$ are adjacent in $T^3$.

Let $P_i$ be a hamiltonian $u-u_i$ path of $T_u^3$ and let $P_j$ be a hamiltonian $v_j-v$ path of $T_v^3$.

The path formed by beginning with $P_i$ followed by the edge $u_i v_j$ and then the path $P_j$ is a hamiltonian $u-v$ path of $T^3$.

Case 2: Suppose that $u$ and $v$ are not adjacent in $T$.

Since $T$ is a tree, there is unique path between every two of its vertices. Let $P$ be the unique $u-v$ path of $T$, and let $f = uw$ be the edge which incident with in $T$.

The graph $T-f$ consists of two trees, $T_u$ and $T_w$ containing $u$ and $w$, respectively.

By the induction hypothesis there exists a hamiltonian $w-v$ path in $P_w$ in $T_u^3$.

Let $u_i$ be a vertex of $T_u$ adjacent to $u$ (or let $u_i = u$ if $T_u$ is trivial), and let $P_i$ be a hamiltonian $u-u_i$ path in $T_u^3$.

Since $d_T(u_i, w) \leq 2$, therefore the edge $u_i w$ is present in $T^3$.

So, the path formed by starting with $P_i$ followed by $u_i w$ and then $P_w$ is a hamiltonian $u-v$ path of $T^3$. 

Rajz a Case 1-hez
Although it is not true that the squares of all connected graphs of order at least 3 are hamiltonian, it was conjectured by Nash-Williams and Plummer that for 2-connected graphs this is the case.

In 1974, Fleischner proved the conjecture to be correct.

**Theorem 5.23.** If \( G \) is 2-connected, then \( G^2 \) is hamiltonian.

A graph \( G \) is \( k \)-(vertex)-connected (\( k \geq 2 \)) if

- either \( K^k \)
- or it has at least \( k+2 \) vertices and no set of \( k-1 \) vertices separates \( G \).

Since \( F \) is 2-connected, then – by **Theorem 5.23** – \( F^3 \) has a hamiltonian cycle \( C \). This cycle contains \( w_1 \) and \( w_2 \).

At least one of the graphs \( G_i \) contains no vertex adjacent to \( w_1 \) or \( w_2 \) on \( C \). Let this graph be \( G_j \).

Since \( u_i \) and \( v_j \) are the only vertices which are adjacent on \( C \) to vertices not in \( G_j \), it follows that \( C \) has a \( u_i-v_j \) path containing all vertices of \( G_j \).

Thus has a hamiltonian \( u_i-v_j \) path, which implies that \( G^2 \) contains a hamiltonian \( u-v \) path.

So, \( G^2 \) is hamiltonian-connected.
The line graph $L(G)$ of a graph $G$ is that graph whose vertices can be put in one-to-one correspondence with the edges of $G$ in such a way that two vertices of $L(G)$ are adjacent iff the corresponding edges of $G$ are adjacent.

In other words: A graph $H$ is a line graph if there exists a graph $G$ such that $H = L(G)$.

It is relatively easy to determine the number of vertices and the number of edges in the line graph $L(G)$ of a graph $G$ in terms of easily computed quantities in $G$:

If given a graph $G(n,m)$ with degree sequence $d_1, d_2, \ldots, d_n$ and $L(G)$ is a graph of order $n'$ and size $m'$ then

$$n' = m \quad \text{and} \quad m' = \sum_{i=1}^{n} \left( \begin{array}{c} d_i \end{array} \right)$$

How can we decide whether a given graph $H$ is a line graph?

Several characterization of line graphs have been obtained, perhaps the best known of which is the “forbidden subgraph” characterisation due to Beineke (1968).

Theorem 5.24.: A graph edge is a line graph iff none of the next graphs is an induced subgraph of $H$. 
A set $X$ of edges in a graph is called a dominating set if every edge of $G$ either belongs to $X$ or is adjacent to an edge of $X$.

If the subgraph induced by the set $X - (X)$ – is a circuit, then $C$ is called dominating circuit of $G$.

A circuit $C$ in a graph $G$ is a dominating circuit iff every edge of $G$ is incident with a vertex of $C$.

The next theorem due to Harary and Nash-Williams provides a characterization of those graphs having hamiltonian line graphs:

**Theorem 5.25.** (F. Harary and J. A. Nash-Williams, 1965): Let $G$ be a graph without isolated vertices. Then $L(G)$ is hamiltonian iff $G=K_{l,l}$ for some $l \geq 3$ or $G$ contains a dominating circuit.

6. Euler Graphs

- Euler trails and euler circuits.
- The seven bridge of Königsberg.
- Eulerian decomposition of graphs.
- Eulerian digraphs
- The exhibition design problem.

**Exercises.** (G. Chartrand and L. Lesniak page 110.)

6.1. Euler circuits

A circuit in a graph $G$ containing all the edges is said to be an Euler circuit of $G$.

An Euler trail contains all edges but it is not closed.

A graph is Eulerian if it has an Euler circuit.
The seven bridges on the Pregel in Königsberg

Theorem 6.1: A non-trivial connected graph has an Euler circuit iff each vertex has even degree.

Proof.

A: \( \iff \) Let \( x_1, x_2, \ldots, x_n \) be an Euler circuit in \( G \). If \( x_i \) occurs \( k \) times in the sequence \( x_1, x_2, \ldots, x_n \), then \( d(x_i) = 2k \).

B: \( \Leftarrow \) By induction on the number of edges.

If there are no edges, then there is nothing to prove, so we proceed the induction step.
Assume we have a graph \( G \) in which each vertex has an even degree. Since \( e(G) \geq 1 \) and \( \delta(G) \geq 2 \) then \( G \) contains a cycle.
(see the Theorem 3.1.)
Let \( C \) be a circuit in \( G \) with the maximal number of edges, and suppose \( C \) is not Eulerian.

Theorem 6.2: A connected graph has an Euler trail from a vertex \( x \) to a vertex \( y \) (\( \neq x \)) iff \( x \) and \( y \) are the only vertices of odd degree.

Proof.

A: \( \iff \) The condition is clearly necessary.

B: \( \Leftarrow \) Suppose that \( x \) and \( y \) are the only vertices of odd degree. Let \( G' \) be obtained from \( G \) by adding to it a vertex \( u \) together with the edges \( ux \) and \( uy \).

By the Theorem 4.3, \( G' \) has an Euler circuit \( C' \). Clearly, \( C' - u \) is an Euler trail from \( x \) to \( y \).
What is the answer for the Kőnigsberg bridge puzzle?

**Theorem 6.3.** If $G$ is a connected graph with $2k$ odd vertices ($k \geq 1$), then the edge set of $G$ can be partitioned into $k$ subsets, each of which induces a trail connecting odd vertices.

Proof is trivial.

**Theorem 6.4.** If $G$ is a connected graph with $2k$ odd vertices ($k \geq 1$), then $E(G)$ can be partitioned into subsets $E_1, E_2, \ldots, E_k$ so that for each $i$, $\langle E_i \rangle$ is a trail connecting odd vertices and such that at most one of these trails has odd length.

6.2. Results on directed graphs.

An eulerian trail of a digraph $D$ is an open trail of $D$ containing all of the arcs and vertices of $D$.

An eulerian circuit is a circuit containing all of the arcs and vertices of $D$.

A digraph that contains an eulerian circuit is called an eulerian digraph.

**Theorem 6.6.** Let $D$ be a nontrivial directed digraph. Then $D$ is eulerian iff $\forall v \in V$, $\deg^+ v = \deg^- v$ for every vertex of $D$.

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**Theorem 6.5.** A nontrivial connected graph $G$ is eulerian iff $E(G)$ can be partitioned into subsets $E_i$, $1 \leq i \leq k$, where each subgraph $\langle E_i \rangle$ is a cycle.

**Proof.**

A: Let $G$ be an eulerian graph. We employ induction on the size of $G$.

- If $m = 3$ then $G = K_3$ and $G$ has the desired property.
- Suppose that the statement is true for any $m' < m$, where $m \geq 4$, and let $G$ be an eulerian graph with $m$ edges.
- Since $G$ is eulerian, so $G$ is an even graph and $G$ has at least one cycle $C$.
- If $E(G) = E(C)$ then we have the desired partition of $E(G)$.

B: Suppose that the edge set of $G$ can be partitioned into subsets $\langle E_i \rangle$, $1 \leq i \leq k$, where each subgraph $\langle E_i \rangle$ is a cycle.

Then $G$ is a nontrivial connected even graph.

We can use, again the Theorem 6.1, and so $G$ is eulerian.
Theorem 6.7.: Let $D$ be a nontrivial connected digraph. Then $D$ has an Eulerian trail if and only if $D$ contains vertices $u$ and $v$ such that

$$od(u) = id(u) + 1 \quad \text{and} \quad id(v) = od(v) + 1$$

and $od(w) = id(w)$ for all other vertices $w$ of $D$. Furthermore, the trail begins at $u$ and ends at $v$.

Another problem from the practice: How shall we plan the corridors of an exhibition?

If the entrance and the exit are the same, a visitor can walk along every corridor once if and only if the corresponding graph has an Eulerian circuit.

In a well-planned exhibition a visitor avoids going along the same corridor twice and continues his walk until there were no new exhibits ahead of him. He chooses his way randomly.

We call the corresponding graph randomly Eulerian.

$G$ is randomly Eulerian from the vertex $x$. 