11. Domination in Graphs

- Some definitions
- Minimal dominating sets
- Bounds for the domination number
- The independent domination number
- Other domination parameters.
“Dominating queens” on the chessboard.

The queen’s graph of the d3 square.

11.2. Minimal Dominating Set.

A minimal dominating set in a graph $G$ is a dominating set that contains no dominating set as a proper subset.

A minimal dominating set of minimum cardinality is a minimum dominating set and consists of $\gamma(G)$ vertices. (See the picture.)

Theorem 11.1: A dominating set $S$ of a graph $G$ is a minimal dominating set of $G$ if and only if every vertex $v$ in $S$ satisfies at least one of the following two properties:

- there exists a vertex $w$ in $V(G) - S$ such that $I(w) \cap S = \{v\}$
- $v$ is adjacent to no vertex of $S$.

Proof.

A: Suppose that $S$ is a minimal dominating set of $G$. Then for each $v \in S$, the set $S - \{v\}$ is not a dominating set of $G$. Consequently, $S$ is a minimal dominating set of $G$.

B: Assume that $S$ is a minimal dominating set of $G$. Then for each $v \in S$, there is a vertex $w$ in $V(G) - (S - \{v\})$ that is adjacent to no vertex of $S - \{v\}$. If $w = v$ then $v$ is adjacent to no vertex of $S$. Therefore, $S$ is a minimal dominating set of $G$. 

“Attacking queens.”

“Nonattacking queens.”
Theorem 11.2: If \( S \) is a minimal dominating set of a graph \( G \) without isolated vertices, then \( V(G) - S \) is a dominating set of \( G \).

Proof.
Let \( v \notin S \). Then \( v \) has at least one of the two properties of the Theorem 11.1.

1. Suppose first that there exists a vertex \( w \) in \( V(G) - S \) such that \( f(w) \cap S = \{ v \} \).
   Hence \( v \) is adjacent to some vertex in \( V(G) - S \).
2. Suppose next that \( v \) is adjacent to no vertex in \( S \).
   Then \( v \) is an isolated vertex of the subgraph \( \langle S \rangle \).
   Since \( v \) is not isolated in \( G \), the vertex \( v \) is adjacent to some vertex of \( V(G) - S \).

Thus \( V(G) - S \) is a dominating set of \( G \).

11.3. Bounds on the Domination Number.

Using the above theorem we have an upper bound for \( \gamma(G) \) in terms of the order of \( G \).

Corollary 11.3: If \( G \) is a graph of order \( n \) without isolated vertices, then \( \gamma(G) \leq n/2 \).

Proof.
Let \( S \) be a minimal dominating set of \( G \). By Theorem 11.2, \( V(G) - S \) is a dominating set of \( G \). Thus
\[
\gamma(G) \leq \min\{ |S|, |V(G) - S| \} \leq n/2.
\]
Theorem 11.6.: If $G$ is a graph of order $n$, then
\[
\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G).
\]

Proof.

We prove first the lower bound.

Let $S$ be a minimum dominating set of $G$. Then

\[ V(G) - S \subseteq \bigcup_{v \in S} \Gamma(v), \]

implying that $|V(G) - S| \leq |S| \cdot \Delta(G)$.

Therefore, $n - \gamma(G) \leq \gamma(G) \cdot \Delta(G)$, and so

\[ \left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G). \]

Theorem 11.8.: If $G$ is a graph without isolated vertices, then

\[ \gamma(G) \leq \min\{\alpha(G), \alpha(G), \beta(G), \beta(G)\}. \]

Proof.

Since every vertex cover of a graph without isolated vertices is a dominating set, as is every maximal independent set of vertices, so

\[ \gamma(G) \leq \alpha(G) \text{ and } \gamma(G) \leq \beta(G). \]

Let $X$ be an edge cover of cardinality $\alpha(G)$. Then every vertex of $G$ is incident with at least one edge in $X$.

Let $S$ be a set of vertices, obtained by selecting an incident vertex with each edge in $X$.

Then $S$ is a dominating set of vertices and $\gamma(G) \leq |S| \leq |X| = \alpha(G)$. 

Without proving we give some further upper bounds:

- If $\delta(G) \geq 1$, then $\gamma(G) \leq n/2$.
- If $\delta(G) \geq 2$ and $G$ is not one of seven exceptional graphs then $\gamma(G) \leq 2n/5$.
- If $\delta(G) \geq 3$ then $\gamma(G) \leq 3n/8$.

**Theorem 11.4.** Let $G$ be a graph of order $n$ with $\delta(G) \geq 2$. Then

\[ \gamma(G) \leq \frac{n}{1 + \ln(\delta + 1)} \]

implies that $V(G) - S \subseteq \bigcup_{v \in S} \Gamma(v)$.

Therefore, $n - \gamma(G) \leq \gamma(G) \cdot \Delta(G)$, and so

\[ \left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G). \]

**Corollary 11.7.** If $G$ is a graph of order $n$, then $\gamma(G) \leq n - \kappa(G)$, where $\kappa(G)$ is the vertex connectivity.

Proof.

The statement follows immediately from the inequality $\kappa(G) \leq \Delta(G)$. 

Now, we establish the upper bound:

Let $v$ be a vertex of $G$ with $\deg v = \Delta(G)$.

Then $V(G) - \Gamma(v)$ is a dominating set of cardinality $n - \Delta(G)$.

So, $\gamma(G) \leq n - \Delta(G)$. 

\[ \text{13} \]

Graph-Theory 9
Let $M$ be a maximum matching in $G$. We construct a set $S$ of vertices consisting of one vertex incident with an edge of $M$ for each edge of $M$.

Let $uv \in M$.

Then, $u$ and $v$ cannot be adjacent to distinct $M$-unsaturated vertices $x$ and $y$, respectively; otherwise $x, u, v, y$ is an $M$-augmenting path in $G$, contradicting Theorem 8.2.

If $u$ is adjacent to an $M$-unsaturated vertex, place $u$ in $S$; otherwise place $v$ in $S$.

This is done for each edge in $M$.

Thus, $S$ is a dominating set of $G$, and $\gamma(G) \leq |S| \leq |M| = \beta_i(G)$.

---

11.4. The Independent Domination Number

A set $S$ of vertices or edges in a graph $G$ is said to be maximal with respect to a property $P$ if $S$ has property $P$ but no proper superset of $S$ has property $P$.

A set $S$ of vertices or edges in a graph $G$ is said to be minimal with respect to a property $P$ if $S$ has property $P$ but no proper subset of $S$ has property $P$.

Example: $K_{s,t}$, where $s < t$, there are two maximal independent sets of vertices: the partite sets of $K_{s,t}$.

A maximal independent set of vertices of maximum cardinality in a graph $G$ is called a maximum independent set of vertices.
Theorem 11.10.: Every maximal independent set of vertices in a graph is a minimal dominating set.

Proof.

Let $S$ be a maximal independent set of vertices in a graph $G$. By Theorem 11.9, $S$ is a dominating set. Since $S$ is independent, certainly every vertex of $S$ is adjacent to no vertex of $S$. Thus, every vertex of $S$ satisfies the second property of Theorem 11.1. So, by Theorem 11.1, $S$ is a minimal dominating set.

Can equality hold? Is there any graph $G$ with $\gamma(G) = i(G)$?

Examples:

* For $K_{1,t}$, $\gamma(K_{1,t}) = i(K_{1,t}) = 1$, for every positive integer $t$.
* For $1 \leq s \leq t$, let $H$ be the graph obtained from $K_{s,t}$ by adding a pendant edge to each vertex of the partite set of cardinality $s$. Then $\gamma(H) = i(H) = s$.
* For the queen's graph $G$, we have $\gamma(G) = i(G) = 5$.

May the difference between the independent domination number and domination number of a graph be arbitrary large?

The double star $T$ containing two vertices of degree $k \geq 2$, where $i(T) = k$ and $\gamma(T) = 2$. 

For some special classes of graphs, Bollobás and Cockayne determined an upper bound for $i(G)$ in terms of $\gamma(G)$.

Theorem 11.11.: If $G$ is a $K_{1,t}$-free graph, where $k \geq 2$, then $i(G) \leq (k - 1)\gamma(G) - (k - 2)$.

Proof.

Let $S$ be a minimum domination set of vertices of $G$ and let $S'$ be a maximal independent set of vertices of $S$ in $G$. Thus, $|S| = \gamma(G)$ and $|S'| \geq 1$.

Now, let $T$ denote the set of all vertices in $V(G) - S$ that are adjacent in $G$ to no vertex of $S'$, and let $T'$ be a maximal independent set of vertices in $T$.

Then, $S' \cup T'$ is an independent set of vertices of $G$. 

$\text{(G)} = i(\text{G})$ ?

$\text{Theorem 11.9. (Berge, 1973): A set } S \text{ of vertices in a graph is an independent dominating set if and only if } S \text{ is maximal independent.}$

Proof.

We have seen already that every maximal independent set of vertices is a dominating set.

Conversely, suppose that $S$ is an independent dominating set. Then $S$ is independent and every vertex not in $S$ is adjacent to a vertex of $S$, so $S$ is maximal independent.

The double star $T$ containing two vertices of degree $k \geq 2$, where $i(T) = k$ and $\gamma(T) = 2$. 

$\text{Theorem 11.11.: If } G \text{ is a } K_{1,t} \text{-free graph, where } k \geq 2, \text{ then } i(G) \leq (k - 1)\gamma(G) - (k - 2).$

Proof.

Let $S$ be a minimum domination set of vertices of $G$ and let $S'$ be a maximal independent set of vertices of $S$ in $G$. Thus, $|S| = \gamma(G)$ and $|S'| \geq 1$.

Now, let $T$ denote the set of all vertices in $V(G) - S$ that are adjacent in $G$ to no vertex of $S'$, and let $T'$ be a maximal independent set of vertices in $T$.

Then, $S' \cup T'$ is an independent set of vertices of $G$. 

$\text{(G)} = i(\text{G})$ ?

$\text{Theorem 11.9. (Berge, 1973): A set } S \text{ of vertices in a graph is an independent dominating set if and only if } S \text{ is maximal independent.}$

Proof.

We have seen already that every maximal independent set of vertices is a dominating set.

Conversely, suppose that $S$ is an independent dominating set. Then $S$ is independent and every vertex not in $S$ is adjacent to a vertex of $S$, so $S$ is maximal independent.
Since every vertex of $V(G) - S'$ is adjacent to some vertex of $S'$, and every vertex of $T - T'$ is adjacent to some vertex of $T'$, it follows that $S' \cup T'$ is a maximal independent set of vertices.

Thus, by Theorem 11.9, $S' \cup T'$ is an independent dominating set.

Observe that every vertex of $S - S'$ is adjacent to at most $k - 1$ vertices of $T'$. (If this were not the case, then some vertex $v$ of $S - S'$ is adjacent to at least $k$ vertices of $T'$, and also at least one vertex of $S'$, which contradicts the hypothesis that $G$ contains no induced subgraph isomorphic to $K_{i,k+1}$.)

Also, observe that every vertex of $T'$ is adjacent to some vertex of $S - S'$.

Therefore

$$|T| \leq (k-1)|S - S'| = (k-1)(|S| - |S'|) = (k-1)(\gamma(G) - |S'|).$$

Consequently,

$$i(G) \leq |S' \cup T'| = |S'| + |T'| \leq |S'| + (k-1)(\gamma(G) - |S'|) = (k-1)(\gamma(G) - (k-2)|S'|) \leq (k-1)(\gamma(G) - (k-2)).$$

11.5. Other Domination Parameters

For a set $A$ of vertices in a graph $G$, the closed neighborhood $N[A]$ of $A$ is defined $N[A] = \cup_{v \in A} N[v]$. (Trivially, $N[A] = I(A) \cup A$.)

A set of vertices in $G$ is called an irredundant set if for every vertex $v \in S$, there exists a vertex $w \in N[v]$ such that $w \notin N[S - \{v\}]$.

Equivalently, $S$ is an irredundant set of vertices if $N[S - \{v\}] \neq N[S]$ for every vertex $v \in S$.

Every vertex $v$ with the property $N[S - \{v\}] \neq N[S]$ is an irredundant vertex.

Consequently, every vertex in an irredundant set is an irredundant vertex.

$S = \{w, y, s\}$ is an irredundant set of vertices.
A set of vertices in a graph $G$ is redundant if there exists a vertex $v \in S$ for which $N[S \setminus \{v\}] \neq N[S]$. Such a vertex $v$ is called a redundant vertex (with respect to $S$).

**Theorem 11.12.** A set $S$ of vertices in a graph $G$ is redundant if every vertex $v$ in $S$ satisfies at least one of the following two properties:

- there exists a vertex $w$ in $V(G) \setminus S$ such that $\Gamma(w) \cap S = \{v\}$
- $v$ is adjacent to no vertex of $S$.

**Proof.**

First, let $S$ be a set of vertices of $G$ such that for every vertex $v \in S$, at least one of the above properties is satisfied.

If the first holds, then there exists a vertex $w \in N[v]$ such that $w \in N[S \setminus \{v\}]$. If the second holds then $v \notin N[S \setminus \{v\}]$.

In either case, $S$ is redundant.

The vertex $v$ may be a private neighbour of itself. Consequently, a nonempty set $S$ of vertices in a graph $G$ is redundant if every vertex of $S$ has a private neighbour.

Certainly every nonempty subset of an irredundant set of vertices in a graph $G$ is irredundant.

Also, every independent set of vertices is an irredundant set.

The irredundant number $ir(G)$ of a graph $G$ is the minimum cardinality among the maximal irredundant sets of vertices of $G$.

Consider the previous picture.

Since $S = \{r, z\}$ is a maximal irredundant set of vertices of minimum cardinality for the graph $G$, it follows that for this graph, $ir(G) = 2$.

Is it true?

Conversely, let $S$ be an irredundant set of vertices in $G$, and let $v \in S$. Since $S$ is irredundant, there exists $w \in N[v]$ such that $w \notin N[S \setminus \{v\}]$.

If $w \neq v$ then the first property is satisfied, if $w = v$ then the second one.

By Theorem 11.1, a minimal dominating set of vertices in a graph is an irredundant set.

Hence, every graph has an irredundant dominating set of vertices.

If $S$ is an irredundant set of vertices in a graph $G$, then for each $v \in S$, the set $N[v] \setminus N[S \setminus \{v\}]$ is nonempty.

Each vertex in $N[v] \setminus N[S \setminus \{v\}]$ is referred to as a private neighbour of $v$.

To see that $S$ is irredundant observe that $y$ is a private neighbour of $r$, and $w$ is a private neighbour of $z$.

To see that $S$ is a maximal irredundant set, note that

- $\{x\}$ is not irredundant since $x$ would have no private neighbour.
- $\{x, r, z\}$ and $\{y, r, z\}$ are not irredundant since $r$ would have no private neighbour.
- $\{x, r, z\}$, $\{v, r, z\}$ and $\{w, r, z\}$ are not irredundant since $z$ would have no private neighbour.

Hence a maximal irredundant set need not be a dominating set and, the irredundance number is not a domination parameter.
Theorem 11.13.: For every graph $G$, 
\[ \text{ir}(G) \leq \gamma(G) \leq i(G). \]

Proof.
We have already observed that $\gamma(G) \leq i(G)$. The inequality $\text{ir}(G) \leq \gamma(G)$ is a consequence of the fact that every minimal dominating set of vertices of $G$ is an irredundant set.

The picture shows that the inequality $\text{ir}(G) \leq \gamma(G)$ may be strict since $\gamma(G) = 3$ and $\text{ir}(G) = 2$.

Example.

\[ \gamma(G) = 3 \text{ and } \text{ir}(G) = 2. \]

The set \{u, v\} is a maximal irredundant set of minimum cardinality in $G$.

To see that \{u, v\} is a maximal irredundant set in $G$, we observe that

* $r$ has no private neighbour in \{t, u, v\},
* $u_1$ has no private neighbour in \{u, v, u_2\} and \{u, v, v_1\},
* $v_1$ has no private neighbour in \{u, v, v_2\} and \{u, v, v_1\},

Cockayne, Favaron, Payan and Thomason (1981) have shown that graphs exist having distinct values for all six parameters mentioned in the previous theorem.

Where can be equality?

\[ \text{Theorem 11.15.: For every bipartite graph } \beta(G) = \gamma(G) = \text{IR}(G). \]

Proof.
Let $G$ be a bipartite graph with partite sets $U$ and $W$. Let $S$ be a maximum irredundant set of vertices in $G$, and let $T$ be the set of isolated vertices of $(S)$. Furthermore, let

\[ U_1 = T \cap U, \quad U_2 = (S \cap U) - T, \]
\[ W_1 = (T \cap W), \quad W_2 = (S \cap W) - T, \]
One or more of these sets may be empty.

Each vertex \( w \in W_2 \) is irredundant in \( S \).

Since \( w \) is not isolated in \( \langle S \rangle \), the vertex \( w \) is not its own private neighbour.

However, since \( S \) is an irredundant set, \( w \) is private neighbour of some vertex of \( V(G) - S \).

Hence for \( w \in W_2 \), there exists a vertex \( w' \in V(G) - S \) such that \( I[w] \cap S = \{w\} \).

Moreover, since \( w \in W \), it follows that \( w' \in U \).

Let \( A = \{w' \mid w \in W_2\} \). Then \( |A| \geq |W_2| \) and \( A \subseteq U \). Furthermore, no vertex of \( A \) is adjacent to a vertex of \( W_2 \).

Consequently, \( U_1 \cup U_2 \cup W_1 \cup A \) is independent in \( G \). Hence

\[
\beta(G) \geq |U_1| + |U_2| + |W_1| + |A| \geq |S| = IR(G).
\]

The result follows from Theorem 11.14.