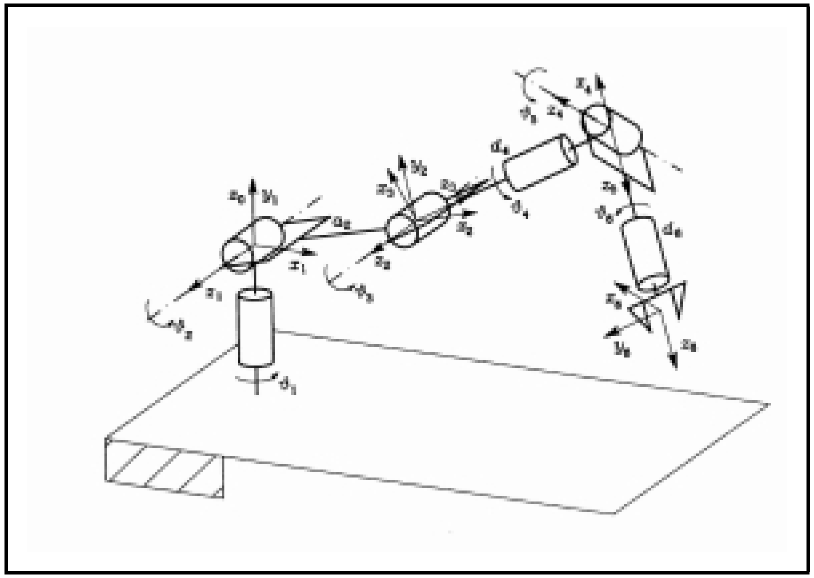
Modelling Serial Manipulators

Modell of *serial* manipulator:

- series of rigid bodies (= *links*)
- connected by means of kinematic pairs (= *joints*).



Rigid bodies can perform ***rigid motions*** $\mathbf{T}$:

- $\mathbf{T}$ preserves distance: $\|\mathbf{T}(\mathbf{q}_1) - \mathbf{T}(\mathbf{q}_2)\| = \|\mathbf{q}_1 - \mathbf{q}_2\|$
- $\mathbf{T}$ preserves orientation: $\mathbf{T}(\mathbf{q}_1) \times \mathbf{T}(\mathbf{q}_2) = \mathbf{q}_1 \times \mathbf{q}_2$

Rigid Motions

have a

- Translational part $\mathbf{p}$.
- and a Rotational part $\mathbf{R}$ with

$$\mathbf{R}^{-1} = \mathbf{R}^T \qquad \det(\mathbf{R}) = 1$$

hence $\mathbf{R} \in \mathcal{SO}(3)$

A rigid body modelled by a set of points $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ now is transformed by

$$\bar{\mathbf{q}}_i = \mathbf{T}(\mathbf{q}_i) = \mathbf{p} + \mathbf{R}\mathbf{q}_i \quad i = 1, \ldots, n$$

Frames and Frame Transformations

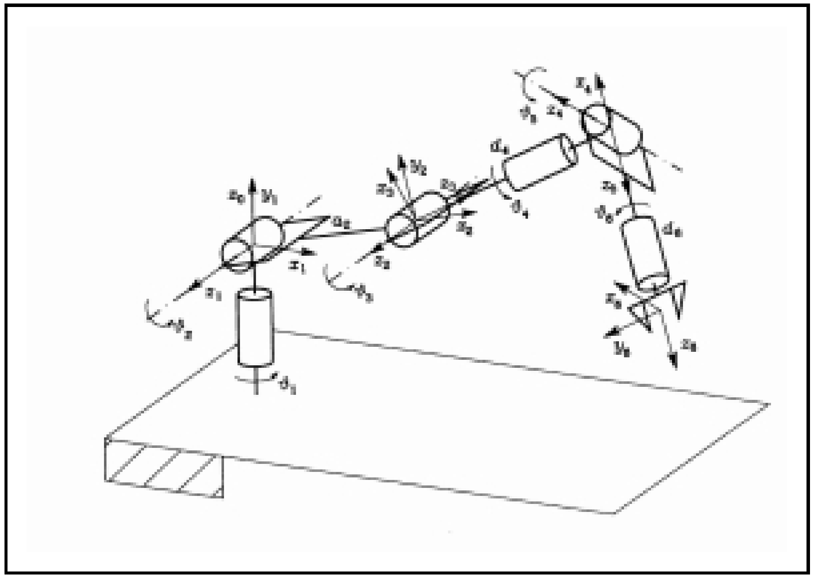
Another view of a **rigid motion** is the transformation of a (coordinate) **frame**:

$$\mathbf{q}_a = \mathbf{T}_{ab}(\mathbf{q}_i) = \mathbf{p}_{ab} + \mathbf{R}_{ab}\mathbf{q}_b$$

where $\mathbf{p}_{ab}, \mathbf{R}_{ab}$ is the specification of the configuration of the B frame relative to the A frame.

Better representation as **Homogeneous Transformations**:

$$\left(\begin{array}{c} \mathbf{q}_a \\ \hline 1 \end{array}\right) = \left(\begin{array}{c|c} \mathbf{R}_{ab} & \mathbf{p}_{ab} \\ \hline \mathbf{0} & 1 \end{array}\right) \left(\begin{array}{c} \mathbf{q}_b \\ \hline 1 \end{array}\right)$$



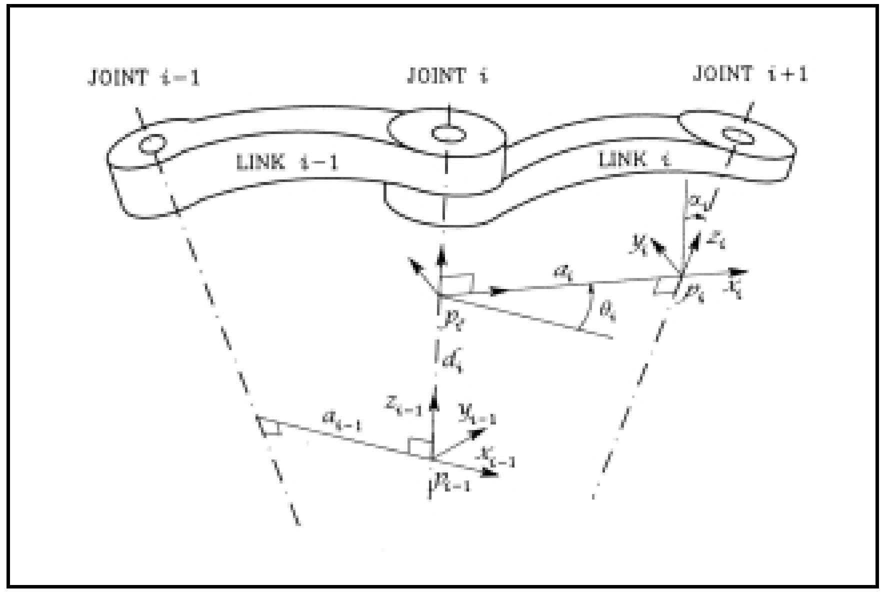
For serial link manipulators the **end-effector frame T** can be displayed by the **closure equation**

$$\mathbf{T} = \mathbf{A}_1 \cdot \mathbf{A}_2 \cdot \mathbf{A}_3 \cdot \mathbf{A}_4 \cdot \mathbf{A}_5 \cdot \mathbf{A}_6$$

where $\mathbf{A}_i$ is the frame attached to the i-th link.

Denavit-Hartenberg parameters

- a_i (*length of link* i)
- d_i (*offset* along joint i)
- α_i (*twist angle* between the axes of joints i and $i+1$)
- θ_i (rotation angle about joint axes i)



This leads to

$$\mathbf{A}_i := \left(\begin{array}{c|c} \mathbf{C}_i \cdot \mathbf{E}_i & \mathbf{C}_i \cdot \mathbf{t}_i \\ \hline \mathbf{0} & 1 \end{array}\right); \quad \mathbf{t}_i = \begin{pmatrix} a_i \\ 0 \\ d_i \end{pmatrix}$$

with

$$\mathbf{C}_i := \begin{pmatrix} c_i & -s_i & 0 \\ s_i & c_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{E}_i := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_i & -\mu_i \\ 0 & \mu_i & \lambda_i \end{pmatrix}$$

while

$c_i := \cos(\theta_i)$, $s_i := \sin(\theta_i)$, $\lambda_i := \cos(\alpha_i)$, $\mu_i := \sin(\alpha_i)$.

Inverse Kinematics

Solve

$$\mathbf{x} = \mathfrak{T}(\theta)$$

for $\theta = (\theta_1, \ldots, \theta_6)^T$. Numerical solution (**Ne ton**)

$$\dot{\theta} = \mathbf{J}^{-1}\,\dot{\mathbf{x}}$$

where **J** is the Jacobi matrix.

PROBLEMS:

- One solution
- dependent to start value

Back to the closure equation

Solving the *inverse kinematics* problem means solving the closure equation

$$\mathbf{T} = \mathbf{A}_1 \cdot \mathbf{A}_2 \cdot \mathbf{A}_3 \cdot \mathbf{A}_4 \cdot \mathbf{A}_5 \cdot \mathbf{A}_6 \tag{1}$$

for the variables $\theta_1, \ldots, \theta_6$.

<u>The problem is nonlinear and has up to 16 solutions.</u>

$$\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 = \hat{\mathbf{A}}_6^{-1}\mathbf{A}_5^{-1}\mathbf{A}_4^{-1} \quad (2)$$

$$\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 = \mathbf{A}_1^{-1}\hat{\mathbf{A}}_6^{-1}\mathbf{A}_5^{-1} \quad (3)$$

$$\mathbf{A}_3\mathbf{A}_4\mathbf{A}_5 = \mathbf{A}_2^{-1}\mathbf{A}_1^{-1}\hat{\mathbf{A}}_6^{-1} \quad (4)$$

$$\mathbf{A}_4\mathbf{A}_5\hat{\mathbf{A}}_6 = \mathbf{A}_3^{-1}\mathbf{A}_2^{-1}\mathbf{A}_1^{-1} \quad (5)$$

$$\mathbf{A}_5\hat{\mathbf{A}}_6\mathbf{A}_1 = \mathbf{A}_4^{-1}\mathbf{A}_3^{-1}\mathbf{A}_2^{-1} \quad (6)$$

$$\hat{\mathbf{A}}_6\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_5^{-1}\mathbf{A}_4^{-1}\mathbf{A}_3^{-1} \quad (7)$$

$$A_{(x)}A_{(a)}A_{(b)} = A_{(c)}^{-1}A_{(d)}^{-1}A_{(e)}^{-1}$$

$$
\begin{aligned}
\mathbf{z}_l &= \mathbf{C}_{(x)}\,\mathcal{A}(\mathbf{C}_{(a)})\,\mathcal{B}(\mathbf{C}_{(b)})\,\mathbf{E}_{(b)}\,\mathbf{e}_3 \\
\mathbf{p}_l &= \mathbf{C}_{(x)}\,\left\{\mathbf{t}_{(x)} + \mathcal{A}(\mathbf{C}_{(a)})\,\mathbf{t}_{(a)} + \mathcal{A}(\mathbf{C}_{(a)})\,\mathcal{B}(\mathbf{C}_{(b)})\,\mathbf{t}_{(b)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{z}_r &= \mathbf{E}^T_{(c)}\,\mathcal{C}(\mathbf{C}^T_{(c)})\,\mathcal{D}(\mathbf{C}^T_{(d)})\,\underbrace{\mathbf{C}^T_{(e)}\,\mathbf{e}_3}_{=\mathbf{e}_3} \\
\mathbf{p}_r &= -\mathbf{E}^T_{(c)}\,\left\{\mathbf{t}_{(c)} + \mathcal{C}(\mathbf{C}^T_{(c)})\,\mathbf{t}_{(d)} + \mathcal{C}(\mathbf{C}^T_{(c)})\,\mathcal{D}(\mathbf{C}^T_{(d)})\,\mathbf{t}_{(e)}\right\}
\end{aligned}
$$

All equations are independent of $\theta_{(e)}$.

Substitute $\theta_{(x)}$ by $x = \tan(\theta_{(x)}/2)$

$$\cos(\theta_{(x)}) = \frac{(1-x^2)}{(1+x^2)}$$

$$\sin(\theta_{(x)}) = \frac{(2x)}{(1+x^2)}$$

This means $\mathcal{X}^{-}\mathbf{C}_{(x)} = \mathcal{X}^{+}$ with

$$\mathcal{X}^{+} := \begin{pmatrix} -x & 1 & 0 \\ 1 & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{X}^{-} := \begin{pmatrix} x & 1 & 0 \\ 1 & -x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 14 Raghavan-and-Roth (RR) equations

$$\mathbf{p}_l = \mathbf{p}_r$$
$$\mathbf{z}_l = \mathbf{z}_r$$
$$\mathbf{p}_l^T \mathbf{p}_l = \mathbf{p}_r^T \mathbf{p}_r$$
$$\mathbf{p}_l^T \mathbf{z}_l = \mathbf{p}_r^T \mathbf{z}_r$$
$$\mathbf{p}_l \times \mathbf{z}_l = \mathbf{p}_r \times \mathbf{z}_r$$
$$\mathbf{p}_l \times (\mathbf{p}_l \times \mathbf{z}_l) + (\mathbf{p}_l^T \mathbf{z}_l)\mathbf{p}_l = \mathbf{p}_r \times (\mathbf{p}_r \times \mathbf{z}_r) + (\mathbf{p}_r^T \mathbf{z}_r)\mathbf{p}_r.$$

These are sufficient to solve the inverse kinematics problem.

The 14 RR-equations

(Use $\mathcal{U} := \mathbf{E}_{(b)}$ and $\mathcal{V} := \mathbf{E}^T_{(c)}$)

$$\mathcal{X}^+\{\mathcal{ABU}\mathbf{e}_3\} = \mathcal{X}^-\mathcal{V}\{\mathcal{CD}\mathbf{e}_3\} \tag{8}$$

$$\left.\begin{array}{r}\mathcal{X}^+\{\mathbf{t}_{(x)}+\mathcal{A}\mathbf{t}_{(a)}+\\ \mathcal{AB}\mathbf{t}_{(b)}\}\end{array}\right\} = \left\{\begin{array}{l}-\mathcal{X}^-\mathcal{V}\{\mathbf{t}_{(c)}+\mathcal{C}\mathbf{t}_{(d)}+\\ \mathcal{CD}\mathbf{t}_{(e)}\}\end{array}\right. \tag{9}$$

$$\left.\begin{array}{r}\mathbf{t}^T_{(x)}\left(\mathbf{t}_{(x)}+2\mathcal{A}\mathbf{t}_{(a)}+2\mathcal{AB}\mathbf{t}_{(b)}\right)\\ +\mathbf{t}^T_{(a)}\left(\mathbf{t}_{(a)}+2\mathcal{B}\mathbf{t}_{(b)}\right)\\ +\mathbf{t}^T_{(b)}\mathbf{t}_{(b)}\end{array}\right\} = \left\{\begin{array}{l}\mathbf{t}^T_{(c)}\left(\mathbf{t}_{(c)}+2\mathcal{C}\mathbf{t}_{(d)}+2\mathcal{CD}\mathbf{t}_{(e)}\right)\\ +\mathbf{t}^T_{(d)}\left(\mathbf{t}_{(d)}+2\mathcal{D}\mathbf{t}_{(e)}\right)\\ +\mathbf{t}^T_{(e)}\mathbf{t}_{(e)}\end{array}\right. \tag{10}$$

$$\left.\begin{array}{r}\mathbf{t}^T_{(x)}\mathcal{ABU}\mathbf{e}_3+\mathbf{t}^T_{(a)}\mathcal{BU}\mathbf{e}_3+\\ \mathbf{t}^T_{(b)}\mathcal{U}\mathbf{e}_3\end{array}\right\} = \left\{\begin{array}{l}-\left(\mathbf{t}^T_{(c)}\mathcal{CD}\mathbf{e}_3+\mathbf{t}^T_{(d)}\mathcal{D}\mathbf{e}_3+\right.\\ \left.\mathbf{t}^T_{(e)}\mathbf{e}_3\right)\end{array}\right. \tag{11}$$

$$\left.\begin{array}{r}\mathcal{X}^{+}\big\{\big(\mathbf{t}_{(x)}\times\mathcal{ABU}\mathbf{e}_3\big)+\\ \mathcal{A}\big(\mathbf{t}_{(a)}\times\mathcal{BU}\mathbf{e}_3\big)+\\ \mathcal{AB}\big(\mathbf{t}_{(b)}\times\mathcal{U}\mathbf{e}_3\big)\big\}\end{array}\right\} = \left\{\begin{array}{l}\mathcal{X}^{-}\mathcal{V}\big\{\big(\mathbf{t}_{(c)}\times\mathcal{CD}\mathbf{e}_3\big)+\\ \mathcal{C}\big(\mathbf{t}_{(d)}\times\mathcal{D}\mathbf{e}_3\big)+\\ \mathcal{CD}\big(\mathbf{t}_{(e)}\times\mathbf{e}_3\big)\big\}\end{array}\right. \tag{12}$$

$$n_1 := \mathbf{t}_{(x)}^T \mathcal{A}\mathcal{B}\mathcal{U}\mathbf{e}_3;\ n_2 := \mathbf{t}_{(a)}^T \mathcal{B}\mathcal{U}\mathbf{e}_3;\ n_3 := \mathbf{t}_{(b)}^T \mathcal{U}\mathbf{e}_3;\ m_1 := \mathbf{t}_{(c)}^T \mathcal{C}\mathcal{D}\mathbf{e}_3;$$

$$m_2 := \mathbf{t}_{(d)}^T \mathcal{D}\mathbf{e}_3;\ m_3 := \mathbf{t}_{(e)}^T \mathbf{e}_3;$$

$$\mathbf{u}_1 := \left(\mathbf{t}_{(x)} \times \mathcal{A}\mathcal{B}\mathcal{U}\mathbf{e}_3\right);\ \mathbf{u}_2 := \left(\mathbf{t}_{(a)} \times \mathcal{B}\mathcal{U}\mathbf{e}_3\right);\ \mathbf{u}_3 := \left(\mathbf{t}_{(b)} \times \mathcal{U}\mathbf{e}_3\right);\ \mathbf{v}_1 := \left(\mathbf{t}_{(c)} \times \mathcal{C}\mathcal{D}\mathbf{e}_3\right);$$

$$\mathbf{v}_2 := \left(\mathbf{t}_{(d)} \times \mathcal{D}\mathbf{e}_3\right);\ \mathbf{v}_3 := \left(\mathbf{t}_{(e)} \times \mathbf{e}_3\right).$$

$$\left.\begin{array}{r} \mathcal{X}^+\Big\{\left(\mathbf{t}_{(x)} \times \mathbf{u}_1\right) + n_1 \mathbf{t}_{(x)} + \\ 2\left[\left(\mathbf{t}_{(x)} \times \mathcal{A}\mathbf{u}_2\right) + n_2 \mathbf{t}_{(x)}\right] + \\ 2\left[\left(\mathbf{t}_{(x)} \times \mathcal{A}\mathcal{B}\mathbf{u}_3\right) + n_3 \mathbf{t}_{(x)}\right] + \\ \mathcal{A}\left(\mathbf{t}_{(a)} \times \mathbf{u}_2\right) + n_2 \mathcal{A}\mathbf{t}_{(a)} + \\ 2\left[\mathcal{A}\left(\mathbf{t}_{(a)} \times \mathcal{B}\mathbf{u}_2\right) + n_3 \mathcal{A}\mathbf{t}_{(a)}\right] + \\ \mathcal{A}\mathcal{B}\left(\mathbf{t}_{(b)} \times \mathbf{u}_3\right) + n_3 \mathcal{A}\mathcal{B}\mathbf{t}_{(b)}\Big\} \end{array}\right\} = \left\{\begin{array}{l} \mathcal{X}^-\mathcal{V}\Big\{\left(\mathbf{t}_{(c)} \times \mathbf{v}_1\right) + m_1 \mathbf{t}_{(c)} + \\ 2\left[\left(\mathbf{t}_{(c)} \times \mathcal{C}\mathbf{v}_2\right) + m_2 \mathbf{t}_{(c)}\right] + \\ 2\left[\left(\mathbf{t}_{(c)} \times \mathcal{C}\mathcal{D}\mathbf{v}_3\right) + m_3 \mathbf{t}_{(c)}\right] + \\ \mathcal{C}\left(\mathbf{t}_{(d)} \times \mathbf{v}_2\right) + m_2 \mathcal{C}\mathbf{t}_{(d)} + \\ 2\left[\mathcal{C}\left(\mathbf{t}_{(d)} \times \mathcal{D}\mathbf{v}_2\right) + m_3 \mathcal{C}\mathbf{t}_{(d)}\right] + \\ \mathcal{C}\mathcal{D}\left(\mathbf{t}_{(e)} \times \mathbf{v}_3\right) + m_3 \mathcal{C}\mathcal{D}\mathbf{t}_{(e)}\Big\} \end{array}\right. \tag{13}$$

Solve an Eigenvalue Problem

Step 1: Transform the 14 RR-equations into the form

$$\boxed{\left(x\hat{\mathcal{L}}+\mathcal{L}\right)\quad {}_{l} = \left(x\hat{\mathcal{R}}+\mathcal{R}\right)\quad {}_{r}} \tag{14}$$

with ($s_a := \sin(\theta_{(a)})$, $c_a := \cos(\theta_{(a)})$, $\ldots$)

$$\begin{aligned} {}_{l} \quad &:= \quad (s_a s_b, s_a c_b, c_a s_b, c_a c_b, s_a, c_a, s_b, c_b, 1)^T \\ {}_{r} \quad &:= \quad (s_c s_d, s_c c_d, c_c s_d, c_c c_d, s_c, c_c, s_d, c_d, 1)^T \end{aligned}$$

and $\hat{\mathcal{L}}$, $\mathcal{L}$, $\hat{\mathcal{R}}$, $\mathcal{R}$ are (14×9)-matrices.

Step 2: (14) can be written as

$$\begin{aligned}(x\mathbf{0} + \mathcal{L}_1) \quad _l &= (x\mathbf{0} + \mathcal{R}_1) \quad _r \\ \left(x\hat{\mathcal{L}}_2 + \mathcal{L}_2\right) \quad _l &= \left(x\hat{\mathcal{R}}_2 + \mathcal{R}_2\right) \quad _r\end{aligned}$$

Step 3: "Rearranging" the columns of the matrices

$$\begin{aligned}(\mathbf{L}_1)\ \hat{}\ _l &= (\mathbf{R}_1)\ \hat{}\ _r \\ \left(x\hat{\mathbf{L}}_2 + \mathbf{L}_2\right)\ \hat{}\ _l &= \left(x\hat{\mathbf{R}}_2 + \mathbf{R}_2\right)\ \hat{}\ _r\end{aligned}$$

with

$$\begin{aligned}\hat{}\ _l &= (s_a s_b, s_a c_b, c_a s_b, c_a c_b, s_a, c_a)^T \\ \hat{}\ _r &= (s_c s_d, s_c c_d, c_c s_d, c_c c_d, s_c, c_c, s_d, c_d, s_b, c_b, 1)^T\end{aligned}$$

Step 4: Perform

$$\mathbf{P} = \mathbf{R}_2 - \mathbf{L}_2\left(\mathbf{L}_1^{-1}\,\mathbf{R}_1\right) \quad \text{and} \quad \hat{\mathbf{P}} = \hat{\mathbf{R}}_2 - \hat{\mathbf{L}}_2\left(\mathbf{L}_1^{-1}\,\mathbf{R}_1\right)$$

to get $\left(x\hat{\mathbf{P}} + \mathbf{P}\right)\,\hat{}_{\,r} = \mathbf{0}$

Step 5: The tangens substitution $t = \tan(\theta_{(c)})$ leads to

$$(x\hat{\mathbf{Q}} + \mathbf{Q})\,\mathbf{q} = \mathbf{0} \tag{15}$$

with $\mathbf{q} = (t^2 s_d, t^2 c_d, t^2, t\,s_d, t\,c_d, t, s_d, c_d, j, k, 1)^T$.

Step 6: By dialytic elimination get

$$(x\mathbf{M}_1 + \mathbf{M}_2)\;\hat{\mathbf{q}} = \mathbf{0} \tag{16}$$

with $\hat{\mathbf{q}} = (t^3 s_d, t^3 c_d, t^3, q_1, \ldots, q_5, t\,j, t\,k, q_6, \ldots, q_{10}, 1)^T$.