Mod p Langlands correspondences via arithmetic geometry

Notes from a Minicourse at KIAS, August 2016 *

Judith Ludwig

January 5, 2017

Lecture 1: Introduction to mod p Langlands correspondences and background on adic spaces

1.1 Mod p Langlands correspondences - Motivation

Let $L \cong \mathbb{C}$ or $L \cong \overline{\mathbb{Q}}_l$. Let $p \neq l$ be a prime number and $F/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_F$, uniformizer ϖ and residue field $\mathbb{F}_q$. Let $G_F := \mathrm{Gal}(\overline{F}/F)$ denote the absolute Galois group of F. The classical local Langlands correspondence for $\mathrm{GL}_n(F)$ is an injection

$$\left\{\begin{array}{l}\text{continuous representations}\\ \rho: G_F \to \mathrm{GL}_n(L)\end{array}\right\}\Big/\cong \;\longrightarrow\; \left\{\begin{array}{l}\text{irreducible smooth}\\ L-\text{representations}\\ \text{ of } \mathrm{GL}_n(F)\end{array}\right\}\Big/\cong$$

$$\rho \longmapsto \pi(\rho)$$

which is characterised by certain identities of L-and ϵ-factors. Enlarging the left side to include all Fobenius-semisimple Weil–Deligne representations, this can be made into a bijection.

We recall some basic definitions.

Definition 1.1. *Let k be a field. A representation $\pi : \mathrm{GL}_n(F) \to \mathrm{Aut}_k(V)$ on a k-vector space V is called* smooth *if for every $v \in V$, the stabilizer*

$$\mathrm{Stab}_{\mathrm{GL}_n(F)}(v) \subset \mathrm{GL}_n(F)$$

*Last modified January 5, 2017. Thanks to Jaclyn Lang and Sug Woo Shin for helpful comments.

is open. A smooth representation π is called admissible *if for every compact open subgroup $K \subset \mathrm{GL}_n(F)$ the subspace*

$$\pi^K := \{v \in V : \pi(k)v = v \ \forall k \in K\}$$

of vectors fixed by K is finite-dimensional.

Let D/F be a division algebra with center F and invariant $1/n$ and let D^* be the group of units in D.

The classical local Jacquet–Langlands correspondence is an injection

$$\left\{\begin{array}{l}\text{irreducible smooth}\\ L-\text{representations}\\ \text{ of } D^*\end{array}\right\}\Big/\cong \quad \longrightarrow \quad \left\{\begin{array}{l}\text{irreducible smooth}\\ L-\text{representations}\\ \text{ of } \mathrm{GL}_n(F)\end{array}\right\}\Big/\cong$$

$$\pi \longmapsto \mathrm{JL}(\pi)$$

which is characterized by a certain equality of traces. There is a maximal compact open subgroup $\mathcal{O}_D^* \subset D^*$, which has the property that $\mathcal{O}_D^* F^* = D^*$, where F^* is identified with the center. This implies that the objects on the left side are finite-dimensional.

Remark 1.2. *Both correspondences can be realized simultanously via geometry; more precisely they can both be realized in the ℓ-adic cohomology of the Lubin–Tate tower, which is a tower of deformation spaces of p-divisible groups. We we will introduce them below.*

Question: Do we have similar correspondences when L is replaced by $\overline{\mathbb{F}}_p$, $\overline{\mathbb{Q}}_p$? Is it possible to understand the mod p and the p-adic cohomology of the Lubin–Tate tower?

In this course we well study mod p correspondences.[1] In [8], Scholze constructed a candidate for the mod p local Langlands and the mod p Jacquet–Langlands correspondence via geometry.

Goal of these lectures: Explain Scholze's construction and study an example.

Before we start with geometry let us remark, that there are two main differences between the theory of smooth admissible representations in characteristic zero and in characterstic p.

[1] A brief remark on the p-adic case: When $L = \overline{\mathbb{Q}}_p$, there are a lot more Galois representations as the topologies of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathrm{GL}_n(\mathbb{Q}_p)$ are more compatible. The category of smooth representations is too small to see all p-adic Galois representations.

- Every non-zero smooth mod p representation of a pro-p group H has non-zero H-fixed vectors.

- In characteristic zero we have the Haar measure as a tool at hand and therefore can use harmonic analysis to study representations. This fails mod p.[2]
  Any $\overline{\mathbb{F}}_p$-valued Haar measure on $\mathrm{GL}_n(F)$ is zero.

  Let's prove this: Let $K(1) := \ker(\mathrm{GL}_n(\mathcal{O}_F) \to \mathrm{GL}_n(\mathbb{F}_q))$. This is a compact open subgroup of $\mathrm{GL}_n(F)$. The subgroup $K(2) := \ker(\mathrm{GL}_n(\mathcal{O}_F) \to \mathrm{GL}_n(\mathcal{O}_F/\varpi^2\mathcal{O}_F))$ has index q^{n^2} in $K(1)$ (the quotient is $K(1)/K(2) \cong M_n(\mathbb{F}_q)$). Any Haar measure μ on $\mathrm{GL}_n(F)$ is translation invariant and in particular $\mu(K(1)) = [K(1) : K(2)]\mu(K(2)) \in \overline{\mathbb{F}}_p$, therefore $\mu(K(1)) = 0$, but $K(1)$ is open, and so μ is zero.

1.2 Crash course on adic spaces

The category - where our geometric objects (like the Lubin–Tate spaces) live - is the category of adic spaces. In this section we briefly review the most important definitions and constructions. One motivation to study adic spaces is that the category of adic spaces encompasses both, formal schemes and rigid spaces.

Definition 1.3. *Let R be a topological ring. A subset $S \subset R$ is called* bounded *if for any open neighbourhood U of zero there exists an open neighbourhood V of zero with $V \cdot S \subset U$. An element $x \in R$ is called* power-bounded *if $\{x^n\}_{n\in\mathbb{N}} \subset R$ is bounded.*

Definition 1.4.

- *A topological ring R is called* Huber *if it contains an open subring $R_0 \subset R$ whose topology is generated by a finitely generated ideal $I \subset R_0$. We call R_0 a* ring of definition *and I an* ideal of definition*.*

- *A* Huber pair *is a pair (R, R^+) where R is a Huber ring and $R^+ \subset R$ is an open and integrally closed subring consisting of power-bounded elements.*

- *A* morphism of Huber pairs $(R, R^+) \to (S, S^+)$ *is a continuous homomorphism $\varphi : R \to S$ such that $\varphi(R^+) \subset S^+$.*

Examples of Huber rings:

- Any ring R with the discrete topology ($R_0 = R$, $I = 0$).

[2] In particular we cannot characterize correspondences in characteristic p as we do in characteristic zero, i.e., via traces.

- Recall that a topological ring R is called *adic* if there exists an ideal $I \subset R$, called *ideal of definition*, s.t. $(I^n)_n$ is a basis of open neighbourhoods of $0 \in R$. Any adic ring R with a finitely generated ideal of definition is a Huber ring.
- For $F/\mathbb{Q}_p$ a complete extension with ring of integers $\mathcal{O}_F$ consider the Tate algebra

  $$F\langle X_1, \cdots, X_n\rangle := \left\{\sum a_{i_1,\cdots,i_n} X_1^{i_1} \cdots X_n^{i_n} \mid a_{i_1,\cdots,i_n} \in F, a_{i_1,\cdots,i_n} \to 0\right\}.$$

  Any quotient of $F\langle X_1, \cdots, X_n\rangle$ is a Huber ring. For $R = F\langle X_1, \cdots, X_n\rangle$, we can take $R_0 = \mathcal{O}_F\langle X_1, \cdots, X_n\rangle$ and $I = (p)$.

Definition 1.5. *A* continuous valuation *on a topological ring R is a map*

$$|\cdot| : R \to \Gamma_{|\cdot|} \cup \{0\},$$

where $\Gamma_{|\cdot|}$ is a multiplicative totally ordered abelian group such that

- $|f \cdot g| = |f| \cdot |g|, \ \forall f, g \in R,$
- $|1| = 1 \in \Gamma_{|\cdot|}, |0| = 0,$
- $|f + g| \leq \max\{|f|, |g|\}, \forall f, g \in R$ *and*
- $\forall \gamma \in \Gamma_{|\cdot|}, \{f \in R : |f| \leq \gamma\} \subseteq R$ *is open.*

We say two continuous valuations $|\cdot|$ and $|\cdot|'$ are equivalent if

$$|a| \leq |b| \iff |a|' \leq |b|'$$

for all $a, b \in R$.

Let (R, R^+) be a Huber pair. Then we define the topological space $X = \mathrm{Spa}(R, R^+)$ as the set

$$X = \mathrm{Spa}(R, R^+) := \{\text{equivalence classes of cts. valuations } x \text{ on } R, \\ \text{s.t. } |f(x)| := x(f) \leq 1 \ \ \forall f \in R^+\}$$

equipped with the topology generated by so called rational subsets

$$X\left(\frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n}\right) := \{x \in X : \forall i, \ |f_i(x)| \leq |s_i(x)| \neq 0 \ \ \forall f_i \in T_i\},$$

where for $i = 1, \ldots, n$, $s_i \in R$, and $T_i \subset R$ is a finite subset which generates an open ideal.

We put a natural structure presheaf $\mathcal{O}_X$ on X as follows.

We equip the ring $R\left[\frac{1}{s_1}, \cdots, \frac{1}{s_n}\right]$ with a topology making $R_0\left[\frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n}\right]$ open equipped with the $J := I \cdot R_0\left[\frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n}\right]$-adic topology. This defines a ring topology on $R\left[\frac{1}{s_1}, \cdots, \frac{1}{s_n}\right]$ and turns it into a Huber ring. Define

$$R\left\langle \frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n} \right\rangle := \text{J-adic completion of } R\left[\frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n}\right].$$

This construction allows us to define structure presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on $X = \mathrm{Spa}(R, R^+)$ by defining them on the basis of rational subsets $U := X\left(\frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n}\right)$ as

$$\mathcal{O}_X(U) := R\left\langle \frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n} \right\rangle$$

and $\mathcal{O}_X^+(U)$ as the completion of the integral closure of $R^+\left[\frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n}\right]$ in $\mathcal{O}_X(U)$.

For each $x \in X$, the valuation $f \mapsto |f(x)|$ on R extends to a valuation $|\cdot|_x$ on the stalk $\mathcal{O}_{X,x}$.

Warning: The presheaf $\mathcal{O}_X$ is not necessarily a sheaf.

Definition 1.6. *A Huber pair* (R, R^+) *is called* sheafy*, if* $\mathcal{O}_X$ *is a sheaf.*

For a Huber ring R, let $R^\circ \subset R$ denote the subring of power-bounded elements.

Example: If $R = F\langle X_1, \cdots, X_n\rangle/\mathfrak{a}$ is a quotient of a Tate algebra, and $R^+ = R^\circ$, then (R, R^+) is sheafy.

Let $\mathcal{V}$ be the category whose objects are triples

$$(X, \mathcal{O}_X, (|\cdot|_x : x \in X)),$$

where X is a topological space, $\mathcal{O}_X$ is a sheaf of complete topological rings on X and $|\cdot|_x$ is an equivalence class of valuations on the stalk $\mathcal{O}_{X,x}$. A morphism from $(X, \mathcal{O}_X, (|\cdot|_x : x \in X))$ to $(Y, \mathcal{O}_Y, (|\cdot|_y : y \in Y))$ is a pair (f, φ) consisting of a continuous map $f : X \to Y$ and a morphism $\varphi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of topological rings such that, for every $x \in X$, the induced ring homomorphism $\varphi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is compatible with the valuations $|\cdot|_x$ and $|\cdot|_{f(x)}$.

Note that any sheafy Huber pair (R, R^+) gives rise to an object

$$\left(\mathrm{Spa}(R, R^+), \mathcal{O}_{\mathrm{Spa}(R,R^+)}, (|\cdot|_x : x \in \mathrm{Spa}(R, R^+))\right) \in \mathcal{V}.$$

Definition 1.7. • *An* affinoid adic space *is an object of $\mathcal{V}$ which is isomorphic to the triple associated with a sheafy Huber pair.*

- *An* adic space *is an object* $(X, \mathcal{O}_X, (|\cdot|_x : x \in X))$ *in $\mathcal{V}$ which is locally an affinoid adic space, i.e., every point $x \in X$ has neighbourhood $U \subset X$ such that* $(U, (\mathcal{O}_X)|_U, (|\cdot|_x : x \in U))$ *is an affinoid adic space.*
- *A* morphism $X \to Y$ of adic spaces *is a morphism in $\mathcal{V}$.*

Remark 1.8. *Below, we often denote an adic space simply by X and we use $|X|$ to refer to its underlying topological space.*

Let K be a complete non-archimedean field.

Theorem 1.9. *There is a fully faithful functor*

$$\begin{aligned} r : \{\textit{rigid analytic varieties}/K\} &\longrightarrow \{\textit{adic spaces} \,/\, \mathrm{Spa}(K, K^\circ)\} \\ \mathrm{Sp}(R) &\longmapsto \mathrm{Spa}(R, R^\circ) \end{aligned}$$

Theorem 1.10. *There is a fully faithful functor*

$$\begin{aligned} t : \{\textit{locally noetherian formal schemes}\} &\longrightarrow \{\textit{adic spaces}\} \\ \mathrm{Spf}(R) &\longmapsto \mathrm{Spa}(R, R) \\ \mathfrak{X} &\longmapsto \mathfrak{X}^{ad} \end{aligned}$$

There is an important construction, the so called *generic fibre construction* that will come up later, so we review it briefly.

Let K be a non-achimedean field, complete with respect to a discrete valuation $|\cdot|_K$. Let $\mathcal{O}_K$ be the ring of integers, ϖ a uniformizer and let k be the residue field. Then $\mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)$ has two points, given by

$$|\cdot|_K : \mathcal{O}_K \to \mathbb{R}_{\geq 0}, a \mapsto |a|_K$$

and

$$|\cdot|_k : \mathcal{O}_K \to \mathcal{O}_K/\varpi \stackrel{|\cdot|_{triv}}{\to} \mathbb{R}_{\geq 0}.$$

The point $|\cdot|_K$ corresponds to a morphism

$$\eta : \mathrm{Spa}(K, \mathcal{O}_K) \to \mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K).$$

The adic generic fibre of a formal scheme $\mathfrak{X}^{\mathrm{ad}}/\mathcal{O}_K$ is defined as

$$\mathfrak{X}^{\mathrm{ad}}_\eta := \mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)} \mathrm{Spa}(K, \mathcal{O}_K).$$

Example: We describe the points of the closed unit disc $X = \mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T\rangle)$. Fix a norm $|\cdot| : \mathbb{C}_p \to \mathbb{R}_{\geq 0}$. The topological space X has five types of points:

- Points of *type (1)*, the *classical points*: Any $x \in \mathcal{O}_{\mathbb{C}_p}$ gives a map

$$\begin{aligned} \mathbb{C}_p\langle T\rangle &\longrightarrow \mathbb{C}_p \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0} \\ f &\mapsto f(x) \longmapsto |f(x)|. \end{aligned}$$

- Points of *type (2) and (3)*: Let $0 \leq r \leq 1$ be a real number, $x \in \mathcal{O}_{\mathbb{C}_p}$. Then

$$|\cdot|_{r,x} : f = \sum a_n (T-x)^n \mapsto \sup_n |a_n| r^n = \sup_{\substack{y \in \mathcal{O}_{\mathbb{C}_p} \\ |y-x| \leq r}} |f(y)| \in \mathbb{R}_{\geq 0}$$

  defines a continuous valuation. We say $|\cdot|_{r,x}$ is of type (2) if $r \in |\mathbb{C}_p^*|$, otherwise we say $|\cdot|_{r,x}$ is of type (3). Note that when $r = 0$, this gives back the classical point x. The valuation $|\cdot|_{r,x}$ depends only on $D(x,r) = \{y \in \mathcal{O}_{\mathbb{C}_p} : |y-x| \leq r\}$. In particular for $r = 1$, $|\cdot|_{1,x}$ is independent of $x \in \mathcal{O}_{\mathbb{C}_p}$; the valuation $|\cdot|_1$ is called the *Gaußpoint*.

- The field $\mathbb{C}_p$ is not spherically complete. Therefore there exists a sequence of discs

$$\mathbb{C}_p \supset D_1 \supset D_2 \supset \cdots$$

  with $\bigcap D_i = \emptyset$. Fix such a sequence, then

$$f \mapsto \inf_i \sup_{x \in D_i} |f(x)| \in \mathbb{R}_{\geq 0}$$

  defines a continuous valuation. We call a point corresponding to such a valuation a point of *type (4)*.

- Finally there are some valuations of rank 2. For that let $x \in \mathcal{O}_{\mathbb{C}_p}$, $0 < r \leq 1$ and choose $* \in \{<, >\}$. Let

$$\Gamma_{*,r} := \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$$

  be the totally ordered abelian group, where $r' < \gamma < r$ for all $r' < r$ if $* =<$ and $r' > \gamma > r$ for all $r' > r$ if $* =>$. Then

$$f = \sum_n a_n (T-x)^n \mapsto \max |a_n| \gamma^n \in \mathbb{R}_{\geq 0} \times \gamma^{\mathbb{Z}}$$

  defines a rank 2 valuation on $\mathbb{C}_p\langle T\rangle$. One can check that if $r \notin |\mathbb{C}_p^*|$, then all these are equivalent to the corresponding point of type (3). But if $r \in |\mathbb{C}_p^*|$ this construction gives new points. They are called points of *type (5)* and they depend only on $D(x, < r) := \{y \in \mathcal{O}_{\mathbb{C}_p} : |y - x| < r\}$ if $* =<$ and on $D(x, r)$ if $* =>$.

One checks that all these valuations are indeed ≤ 1 on $\mathcal{O}_{\mathbb{C}_p}\langle T\rangle$. All points except those of type (2) are closed. A point of type (2) has points of type (5) in its closure.

Lecture 2: Perfectoid spaces and Scholze's functor

2.1 Scholze's functor

Let $F/\mathbb{Q}_p$ be a finite extension. Let D be a division algebra over F with center F and invariant $1/n$. In [8], Scholze constructs a functor

$$\begin{aligned} \mathcal{S} : \left\{ \begin{array}{l} \text{smooth admissible} \\ \mathbb{F}_p\text{-representations} \\ \text{of}\, \mathrm{GL}_n(F) \end{array} \right\} &\longrightarrow \left\{ \begin{array}{l} \text{smooth admissible} \\ \mathbb{F}_p\text{-representations} \\ \text{of } D^* \end{array} \right\} \\ \pi &\longmapsto \mathcal{S}(\pi). \end{aligned}$$

The space $\mathcal{S}(\pi)$ also carries an action of $\mathrm{Gal}(\overline{F}/F)$.

Goal of this lecture: Explain his construction.

2.2 Perfectoid spaces

As the construction of $\mathcal{S}$ involves the Lubin–Tate space at infinite level, which is a perfectiod space, I will give the definition of a perfectoid space next. For a Huber ring R, recall that R° denotes the subset of power-bounded elements.

Definition 2.1. *A perfectoid field is a non-archimedean field K of residue characteristic $p > 0$, complete with respect to a rank 1 non-discrete valuation such that*

$$\mathrm{Frob} : K^\circ/\varpi \to K^\circ/\varpi, x \mapsto x^p$$

is surjective, where $\varpi \in K^*$ is a topologically nilpotent unit.

Examples: $\mathbb{C}_p$, $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ and $\overline{\mathbb{F}}_p((t^{1/p^\infty}))$ are examples of perfectoid fields. Non-examples are $\mathbb{Q}_p$ and $\mathbb{F}_p$.

Fix a perfectoid field K.

Definition 2.2. *A* perfectoid K-algebra *is a Banach algebra R such that $R^\circ \subset R$ is open and bounded and*

$$\mathrm{Frob} : R^\circ/\varpi \to R^\circ/\varpi, x \mapsto x^p$$

is surjective.

Note that by definition the subring R° of a perfectoid algebra R is open and bounded. Therefore R° can serve as a ring of definition.

Definition 2.3. *An* affinoid perfectoid K-algebra *is a Huber pair* (R, R^+) *such that* R *is a perfectoid* K*-algebra.*

For an affinoid perfectoid K-algebra, the presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are always sheaves.[3] The reason for this is the boundedness of R° in R and the presence of a topologically nilpotent unit. Here's the general picture:

Definition 2.4.
1. *A Huber ring is called* Tate *if it contains a topologically nilpotent unit.*
2. *A Huber ring* R *is called* uniform, *if* $R^\circ \subset R$ *is bounded.*
3. *A Huber pair* (R, R^+) *is called* stably uniform, *if* $\mathcal{O}_X(U)$ *is uniform for all rational subsets* $U \subseteq X = \mathrm{Spa}(R, R^+)$.

Exercise: Show that $\mathbb{Q}_p[T]/(T^2)$ is not uniform.

Theorem 2.5 ([1, Theorem 7]). *If* (R, R^+) *is a stably uniform Huber pair, with* R *Tate, then* $\mathcal{O}_{\mathrm{Spa}(R,R^+)}$ *is a sheaf.*

Theorem 2.6 ([6, Theorem 6.3]). *If* (R, R^+) *is affinoid perfectoid,* $U \subset X = \mathrm{Spa}(R, R^+)$ *rational, then* $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ *is affinoid perfectoid. In particular* $\mathcal{O}_X$ *is a sheaf, so* $X = \mathrm{Spa}(R, R^+)$ *is an adic space.*

As before fix a perfectoid field K.

Definition 2.7. *An* affinoid perfectoid space *over* K *is an affinoid adic space* $\mathrm{Spa}(R, R^+)$ *over* K *with* R *a perfectoid algebra. A* perfectoid space *is an adic space* X *over a* K *that has a cover by affinoid perfectoid spaces. Morphisms between perfectoid spaces are the morphisms of adic spaces.*

One word of caution: If X/K is a perfectoid space and $\mathrm{Spa}(R, R^+) \subset X$ is an affinoid open, then we don't know whether R is a perfectoid K-algebra.

2.3 The Lubin–Tate tower

As we recall in this section, the Lubin–Tate tower is a tower $\mathcal{M}_n$ of rigid analytic varieties parametrizing deformations of a p-divisible group together with some level structure. The inverse limit $\varprojlim \mathcal{M}_n$ can be equipped with the structure of a perfectoid space.

Let $F = \mathbb{Q}_p$, $k = \overline{\mathbb{F}}_p$ and fix a connected p-divisible group H/k of dimension one and height n. Then $D = \mathrm{End}(H)$ is a division algebra with center $\mathbb{Q}_p$ and invariant $1/n$.

[3] It suffices to check the sheaf property for $\mathcal{O}_X$, as if $\mathcal{O}_X$ is a sheaf, then this implies that $\mathcal{O}_X^+$ is also a sheaf.

Definition 2.8. *Let* $\mathrm{Nilp}_{W(k)}$ *be the category of* $W(k)$*-algebras* R *in which* p *is nilpotent. A* deformation *of* H *to* $R \in \mathrm{Nilp}_{W(k)}$ *is a pair* (G, ρ) *where* G *is a* p*-divisible group over* R *and*

$$\rho : H \otimes_k R/p \to G \otimes_R R/p$$

is a quasi-isogeny.

Let $\mathrm{Def_H}$ be the functor

$$\begin{aligned} \mathrm{Def_H} : \mathrm{Nilp}_{W(k)} &\to \mathit{Sets} \\ R &\mapsto \{(G,\rho) : \text{ deformation of } H \text{ to } R\}/_{\cong}. \end{aligned}$$

Theorem 2.9 ([5])**.** *The functor* Def_H *is representable by a formal scheme* $\mathfrak{M}/W(k)$*. We have a decomposition*

$$\mathfrak{M} \cong \bigsqcup_{i \in \mathbb{Z}} \mathfrak{M}^{(i)}$$

according to the height i *of the quasi-isogeny and non-canonically*

$$\mathfrak{M}^{(i)} \cong \mathrm{Spf}(W(k)[[t_1, \dots, t_{n-1}]]).$$

Let $\mathcal{M}_0 := \mathfrak{M}_\eta^{ad} \times_{\breve{\mathbb{Q}}_p} \mathbb{C}_p$ be the (base change to $\mathbb{C}_p$ of the) adic generic fibre of $\mathfrak{M}$. One can introduce level structures to get spaces $\mathcal{M}_m$ for any integer $m \geq 0$, as well as finite étale maps $\mathcal{M}_m \to \mathcal{M}_0$.

The tower $(\mathcal{M}_m)_m$ can be equipped with the structure of a perfectoid space. In order to express this we need the following notion.

Definition 2.10. *[cf. Def. 2.4.1 in [9]] Let* $(X_m)_{m \in I}$ *be a cofiltered inverse system of adic spaces with finite étale transition maps. Let* X *be an adic space. Write*

$$X \sim \varprojlim X_m$$

if there exists a compatible family of maps $X \to X_m$ *such that*

- *the map of underlying topological spaces* $|X| \to \varprojlim |X_m|$ *is a homeomorphism and*
- *there is an open cover of* X *by affinoid subsets* $\mathrm{Spa}(R, R^+) \subset X$ *such that the map*

$$\varinjlim_{\mathrm{Spa}(R_m, R_m^+) \subset X_m} R_m \to R$$

  has dense image, where the direct limit runs over all open affinoids $\mathrm{Spa}(R_m, R_m^+) \subset X_m$ *over which* $\mathrm{Spa}(R, R^+) \subset X \to X_m$ *factors.*

Theorem 2.11 ([9, 6.3.4]). *There exists a unique up to unique isomorphism perfectoid space $\mathcal{M}_\infty$ over $\mathbb{C}_p$ such that*

$$\mathcal{M}_\infty \sim \varprojlim \mathcal{M}_m.$$

The infinite-level Lubin–Tate space $\mathcal{M}_\infty$ represents the functor from complete affinoid $(W(k)[\frac{1}{p}], W(k))$-algebras to *Sets* which sends (R, R^+) to the set of triples (G, ρ, α) where $(G, \rho) \in \mathcal{M}_0(R, R^+)$ and

$$\alpha : \mathbb{Z}_p^2 \to T(G)_\eta^{ad}(R, R^+)$$

is a morphism of $\mathbb{Z}_p$-modules such that for all points $x = \mathrm{Spa}(K, K^+) \in \mathrm{Spa}(R, R^+)$, the induced map

$$\alpha(x) : \mathbb{Z}_p^2 \to T(G)_\eta^{ad}(K, K^+)$$

is an isomorphism, cf. [9, Section 6.3]. Here $T(G)$ is the integral Tate module, i.e., the sheaf on Nilp_R^{op} defined as $T(G)(S) = \varprojlim G[p^n](S)$, and $T(G)_\eta^{ad}$ is the adic generic fibre (over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$).

The infinite-level Lubin–Tate space $\mathcal{M}_\infty$ has commuting actions of the group D^* (which acts via its action on H), of the group $\mathrm{GL}_n(\mathbb{Q}_p)$ and of the Weil group $W_{\mathbb{Q}_p}$.

We have a decomposition into perfectoid spaces

$$\mathcal{M}_\infty \cong \bigsqcup_{i \in \mathbb{Z}} \mathcal{M}_\infty^{(i)}$$

at infinite level and for each $i \in \mathbb{Z}$ the stabilizer $\mathrm{Stab}_{\mathrm{GL}_n(\mathbb{Q}_p)}(\mathcal{M}_\infty^{(i)})$ is given by the subgroup

$$G' := \{g \in \mathrm{GL}_n(\mathbb{Q}_p) | \ \det(g) \in \mathbb{Z}_p^*\}.$$

If $F/\mathbb{Q}_p$ is a finite extension we have similar deformation spaces of p-divisible groups with an $\mathcal{O}_F$-action, and in the limit again a perfectoid space $\mathcal{M}_\infty$ with actions of $\mathrm{GL}_n(F), D^*/F$ and W_F.

Remark 2.12. *The classical local Langlands correspondence and the classical Jacquet–Langlands correspondence can be realized in the l-adic cohomology of the Lubin–Tate tower. As we would like to understand mod p correspondences, it seems that it could be a good idea to study the mod p cohomology of $\mathcal{M}_\infty$. Unfortunately the cohomology groups $H^i_{\text{ét}}(\mathcal{M}_\infty, \mathbb{F}_p)$ are not at all well behaved, e.g., they depend on the choice of a complete algebraic closure of $\breve{F}$ and they are not admissible (see [2]).*

In order to get a candidate for a mod p local Langlands correspondence and a mod p Jacquet–Langlands correspondence, Scholze uses the Gross–Hopkins period morphism

$$\pi_{\mathrm{GH}} : \mathcal{M}_\infty \to \mathbb{P}^{n-1}_{\mathbb{C}_p}.$$

- This is a surjective map.
- It is $\mathrm{GL}_n(F)$ equivariant (where the action on $\mathbb{P}^{n-1}_{\mathbb{C}_p}$ is the trivial action).
- It is also equivariant for the D^* action (via the natural action of $D^*(\breve{F}) \cong \mathrm{GL}_n(\breve{F})$ on $\mathbb{P}^{n-1}_{\breve{F}} \times_{\breve{F}} \mathbb{C}_p^*$)
- It factors through a corresponding map at all finite levels $\pi_{\mathrm{GH},m} : \mathcal{M}_m \to \mathbb{P}^{n-1}_{\mathbb{C}_p}$ and all these are étale covering maps.

Note that the existence of π_{GH} implies that the analytic projective space $\mathbb{P}^{n-1}_{\mathbb{C}_p}$ is very far from being simply connected.

2.4 Definition of Scholze's functor

We now have all the prerequisites to define Scholze's functor $\mathcal{S}$.

Let π be an admissible smooth representation of $\mathrm{GL}_n(F)$ on an $\mathbb{F}_p$-vector space. For an étale map $U \to \mathbb{P}^{n-1}_{\mathbb{C}_p}$ define

$$\mathcal{F}_\pi(U) := \mathrm{Map}_{\mathrm{cont},\mathrm{GL}_n(F)}(|U \times_{\mathbb{P}^{n-1}} \mathcal{M}_\infty|, \pi).$$

Here one turns the right action of $\mathrm{GL}_n(F)$ on $\mathcal{M}_\infty$ into a left action and then the subscript ${}_{\mathrm{GL}_n(F)}$ means $\mathrm{GL}_n(F)$-equivariant maps.

By [8, Proposition 3.1], this defines a sheaf $\mathcal{F}_\pi$ on $(\mathbb{P}^{n-1}_{\mathbb{C}_p})_{\text{ét}}$. Moreover the functor that sends π to $\mathcal{F}_\pi$ is an exact functor.

Theorem 2.13 ([8, Theorem 1.1])**.** *For any admissible smooth representation π of $\mathrm{GL}_n(F)$, the cohomology groups*

$$\mathcal{S}^i(\pi) := H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi), i \geq 0,$$

are admissible D^*-representations. They carry an action of $\mathrm{Gal}(\overline{F}/F)$ and they vanish for all $i > 2(n-1)$.

Lecture 3: The case of a principal series representation of $\mathrm{GL}_2(\mathbb{Q}_p)$

In this lecture we study Scholze's functor in the case where $F = \mathbb{Q}_p$, $n = 2$ and π is a principal series representation.

Recall that we have an exact functor

$$\left\{\begin{array}{l}\text{smooth admissible}\\ \mathbb{F}_p\text{-representations}\\ \text{of}\,\mathrm{GL}_2(\mathbb{Q}_p)\end{array}\right\} \longrightarrow \left\{\text{sheaves on } (\mathbb{P}^1_{\mathbb{C}_p})_{\text{ét}}\right\}$$
$$\pi \longmapsto \mathcal{F}_\pi.$$

The cohomology groups $\mathcal{S}^i(\pi) := H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_\pi)$ are admissible D^*-representations which carry a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Furthermore $\mathcal{S}^i(\pi) = 0$ for all $i > 2$.

One can always easily compute $\mathcal{S}^0(\pi)$. For a representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ let $\pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$ denote the space of vectors that are invariant under $\mathrm{SL}_2(\mathbb{Q}_p)$. This is a subrepresentation of π.

Proposition 3.1 ([8, Prop. 4.7]). *The natural map*

$$H^0_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\pi^{\mathrm{SL}_2(\mathbb{Q}_p)}}) \hookrightarrow H^0_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_\pi)$$

is an isomorphism.

Note that if π is irreducible and $\pi^{\mathrm{SL}_2(\mathbb{Q}_p)} \neq 0$, then π is finite-dimensional. But the only irreducible finite-dimensional smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ are the one-dimensional representations, so we see that

$$\mathcal{S}^0(\pi) = 0,$$

whenever π is irreducible and not one-dimensional.

We now want to study $\mathcal{S}^2(\pi)$ when π is a principal series representation. For that let $B(\mathbb{Q}_p) \subset \mathrm{GL}_2(\mathbb{Q}_p)$ denote the Borel subgroup of upper triangular matrices. Let $q = p^m$ for some $m \geq 1$. Let $\chi_i : \mathbb{Q}_p^* \to \mathbb{F}_q^*, i = 1, 2$ be two smooth characters. They give rise to a character $\chi = (\chi_1, \chi_2)$ of the Borel subgroup via

$$b = \begin{pmatrix} b_1 & * \\ & b_2 \end{pmatrix} \mapsto \chi_1(b_1)\chi_2(b_2).$$

The principal series representation $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ is defined as

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi) := \left\{ f : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{F}_q : \begin{array}{l} f \text{ locally constant and} \\ f(bg) = \chi(b) f(g) \forall b \in B(\mathbb{Q}_p), g \in \mathrm{GL}_2(\mathbb{Q}_p) \end{array} \right\},$$

where $\mathrm{GL}_2(\mathbb{Q}_p)$ acts via right translation. The representation $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi)$ is irreducible if and only if $\chi_1 \neq \chi_2$.

In the rest of this lecture we'll explain some ideas behind the proof of the following theorem.

Theorem 3.2 ([4, Theorem 4.6])**.** *Let $\pi := \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi)$ be a principal series representation, where $\chi = (\chi_1, \chi_2)$, and $\chi_i : \mathbb{Q}_p^* \to \mathbb{F}_q^*$ are smooth characters. Then $\mathcal{S}^2(\pi) = 0$.*

Remark 3.3. *The theorem and Proposition 3.1 imply that, for $\chi_1 \neq \chi_2$, the cohomology $\mathcal{S}^i(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi))$ is concentrated in the middle degree.*

The proof of the theorem can be divided into two steps.

- **Step 1**: One constructs a quotient $\mathcal{M}_\infty / B(\mathbb{Q}_p)$ in the category of perfectoid spaces over $\mathbb{C}_p$.
- **Step 2**: The Gross–Hopkins period morphism π_{GH} factors through the quotient $\mathcal{M}_\infty / B(\mathbb{Q}_p)$. Let $\overline{\pi}_{\mathrm{GH}} : \mathcal{M}_\infty / B(\mathbb{Q}_p) \to \mathbb{P}^1_{\mathbb{C}_p}$ be the induced map. This map has many nice properties, in particular, it is quasi-compact. One can show that $\mathcal{F}_\pi \cong \overline{\pi}_{\mathrm{GH},*}\mathcal{F}_\chi$ for a finite rank local system $\mathcal{F}_\chi$ on $\mathcal{M}_\infty / B(\mathbb{Q}_p)$, which is defined in terms of χ. Then one uses that $\mathcal{M}_\infty / B(\mathbb{Q}_p)$ is perfectoid and the nice properties of $\overline{\pi}_{\mathrm{GH}}$ to deduce vanishing of $\mathcal{S}^2(\pi)$.

Below, we will focus on explaining the ideas behind the construction of the quotient in Step 1.

For Step 2 let us just remark that once one has constructed the perfectoid space $\mathcal{M}_\infty / B(\mathbb{Q}_p)$ one defines the local system $\mathcal{F}_\chi$ on $\mathcal{M}_\infty / B(\mathbb{Q}_p)$ as follows.

Let $\chi = (\chi_1, \chi_2), \chi_i : \mathbb{Q}_p^* \to \mathbb{F}_q^*$, be a pair of smooth characters considered as a representation of the Borel subgroup $B(\mathbb{Q}_p)$. For $U \overset{\text{ét}}{\to} \mathcal{M}_\infty / B(\mathbb{Q}_p)$ let

$$\mathcal{F}_\chi(U) := \mathrm{Map}_{\mathrm{cont}, B(\mathbb{Q}_p)}(|U \times_{\mathcal{M}_\infty / B(\mathbb{Q}_p)} \mathcal{M}_\infty, \chi).$$

This defines an étale local system of rank 1 over $\mathbb{F}_q$ on $\mathcal{M}_\infty / B(\mathbb{Q}_p)$.

The Gross–Hopkins period morphism $\pi_{\mathrm{GH}} : \mathcal{M}_\infty \to \mathbb{P}^1_{\mathbb{C}_p}$ factors through the quotient $\mathcal{M}_\infty / B(\mathbb{Q}_p)$ and we get an induced map

$$\overline{\pi}_{\mathrm{GH}} : \mathcal{M}_\infty / B(\mathbb{Q}_p) \to \mathbb{P}^1_{\mathbb{C}_p},$$

which turns out to be quasi-compact.

One can show that $\overline{\pi}_{\mathrm{GH},*}\mathcal{F}_\chi = \mathcal{F}_\pi$ and that the higher direct images all vanish,

$$R^i\overline{\pi}_{\mathrm{GH},*}\mathcal{F}_\chi = 0,\ \forall i > 0.$$

Hence

$$\mathcal{S}^i\left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi)\right) = H^i_{\text{ét}}(\mathcal{M}_\infty/B(\mathbb{Q}_p), \mathcal{F}_\chi).$$

We see that at the expense of a more complicated space, the sheaf has simplified a great deal. And although the space is more complicated, it is perfectoid. This fact and the good properties of $\overline{\pi}_{\mathrm{GH}}$ make it possible to show vanishing of, well, not directly of $H^2_{\text{ét}}(\mathcal{M}_\infty/B(\mathbb{Q}_p), \mathcal{F}_\chi)$, but instead of $H^2_{\text{ét}}(\mathcal{M}_\infty/B(\mathbb{Q}_p), \mathcal{F}_\chi \otimes \mathcal{O}^+/p)$. It turns out that this suffices to deduce vanishing of $\mathcal{S}^2\left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi)\right)$. For details regading these cohomological calculations we refer to [4].

3.1 The quotient $\mathcal{M}_\infty/B(\mathbb{Q}_p)$

First, let us mention a general result on quotients.

Proposition 3.4. *Let X be a rigid analytic space (resp. a perfectoid space defined over a perfectoid field of characteristic zero). Let G be a finite group which acts on X. Assume that X has a cover by G-stable affinoid open subsets* $\mathrm{Spa}(R, R^+)$*. Then X/G exists as a rigid space (resp. a perfectoid space). The quotient* $\mathrm{Spa}(R, R^+)/G$ *is affinoid and given by* $\mathrm{Spa}(R^G, R^{+G})$.

Proof. See Theorem 1.1, Theorem 1.2 and Theorem 1.4 in [3]. □

The space $\mathcal{M}_\infty/B(\mathbb{Q}_p)$ is constructed in two steps. Recall that $\mathcal{M}_\infty = \bigsqcup_i \mathcal{M}_\infty^{(i)}$. First one constructs the quotient $\mathcal{M}_\infty^{(0)}/B(\mathbb{Z}_p)$. In a second step one constructs $\mathcal{M}_\infty/B(\mathbb{Q}_p)$ from $\mathcal{M}_\infty^{(0)}/B(\mathbb{Z}_p)$.

Theorem 3.5 ([4, Theorem 3.4])**.** *There exists a perfectoid space $\mathcal{M}_\infty^{(0)}/B(\mathbb{Z}_p)$ together with compatible maps*

$$\mathcal{M}_\infty^{(0)}/B(\mathbb{Z}_p) \to \mathcal{M}_m^{(0)}/B(\mathbb{Z}/p^m\mathbb{Z})$$

such that

$$\mathcal{M}_\infty^{(0)}/B(\mathbb{Z}_p) \sim \varprojlim_m \mathcal{M}_m^{(0)}/B(\mathbb{Z}/p^m\mathbb{Z}).$$

The space $\mathcal{M}_\infty^{(0)}/B(\mathbb{Z}_p)$ is the quotient of $\mathcal{M}_\infty^{(0)}$ by $B(\mathbb{Z}_p)$ in the category of perfectoid spaces.

The space $\mathcal{M}_\infty^{(0)}/B(\mathbb{Z}_p)$ is constructed using ∞-level modular curves. In [7], Scholze has shown that the ∞-level modular curve

$$\mathcal{X}^*_{\Gamma(p^\infty)} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}$$

is a perfectoid space. It has an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ and there is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant period map, the so called Hodge–Tate period morphism

$$\pi_{\mathrm{HT}} : \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathbb{P}^1_{\mathbb{C}_p}$$

where the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathbb{P}^1_{\mathbb{C}_p}$ is the usual left action turned into a right action, i.e., it is given by

$$[x_0 : x_1] \begin{pmatrix} a & b \\ c & d \end{pmatrix} = [dx_0 - bx_1 : -cx_0 + ax_1].$$

We have a decomposition

$$\begin{aligned} \mathcal{X}^*_{\Gamma(p^\infty)} &= \mathcal{X}^{\mathrm{ord}}_{\Gamma(p^\infty)} \sqcup \mathcal{X}^{ss}_{\Gamma(p^\infty)} \\ &= \pi^{-1}_{\mathrm{HT}}(\mathbb{P}^1(\mathbb{Q}_p)) \sqcup \pi^{-1}_{\mathrm{HT}}(\Omega^2), \end{aligned}$$

where Ω^2 denotes the Drinfeld upper half plane $\Omega^2 = \mathbb{P}^1 \backslash \mathbb{P}^1(\mathbb{Q}_p)$.

The Lubin–Tate tower sits inside the ∞-level modular curve, in fact

$$\mathcal{X}^{ss}_{\Gamma(p^\infty)} = \bigsqcup_{\text{finite}} \mathcal{M}_\infty^{(0)}.$$

Fix an embedding $\iota : \mathcal{M}_\infty^{(0)} \hookrightarrow \mathcal{X}^{ss}_{\Gamma(p^\infty)}$.

Furthermore "part of the tower $\left(\mathcal{X}^*_{\Gamma_0(p^m)}\right)_m$" is perfectoid. More precisely there is a family of open subspaces[4]

$$\mathcal{X}^*_{\Gamma_0(p^m)}(\epsilon)_a \subset \mathcal{X}^*_{\Gamma_0(p^m)},\ \ 0 \le \epsilon < 1/2,$$

that in the limit give a perfectoid space

$$\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \sim \varprojlim \mathcal{X}^*_{\Gamma_0(p^m)}(\epsilon)_a.$$

There is a corresponding open $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \subset \mathcal{X}^*_{\Gamma(p^\infty)}$ and an open map

$$\phi : \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a. \tag{1}$$

[4] Roughly speaking: $\mathcal{X}^*_{\Gamma_0(p^m)}(\epsilon)_a$ is the locus where the subgroup is anticanonical and the Hasse invariant satisfies $|Ha| \ge |p|^{p^{-m}\epsilon}$.

In fact (and very importantly for us) the spaces $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ and $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ are affinoid perfectoid. Finally

$$\mathcal{M}_\infty \subset \bigcup_{i=0}^{\infty} \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \cdot \gamma^i,$$

where $\gamma = \begin{pmatrix} p & \\ & p^{-1} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$. We can build a candidate for the quotient $\mathcal{M}^{(0)}_\infty / B(\mathbb{Z}_p)$ using the open map ϕ from (1).

Before we explain this, let us mention a technical annoyance regarding inverse limits of adic spaces: When X is an adic space, which is a limit of a cofiltered system $(X_m)_m$ by which we mean that

$$X \sim \varprojlim X_m,$$

then unfortunately we do not know whether an arbitrary affinoid open

$$U = \mathrm{Spa}(R, R^+) \subset X$$

is the preimage $p_m^{-1}(U_m)$ of an open affinoid subset $U_m = \mathrm{Spa}(R_m, R_m^+) \subset X_m$ for sufficiently large m and if it satisfies $R^+ \cong (\varinjlim R_m^+)^\wedge$. For this reason some of the arguments below become slightly technical and to facilitate notation we introduce the following notion.

Definition 3.6. *Let $(K, \mathcal{O}_K)$ be a non-archimedean field. Let*

$$(X_m = \mathrm{Spa}(R_m, R_m^+))_{m \in I}$$

be a cofiltered inverse system of affinoid adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$ with finite étale transition maps. Assume $X = \mathrm{Spa}(R, R^+)$ is an affinoid adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$ with a compatible family of maps $p_m : X \to X_m$. We write

$$X \approx \varprojlim X_m$$

if R^+ is the ϖ-adic completion of $\varinjlim_m R_m^+$.

Obviously if $X \approx \varprojlim X_m$ then also $X \sim \varprojlim X_m$.

In order to construct the quotient $\mathcal{M}^{(0)}_\infty / B(\mathbb{Z}_p)$ we first show that every point in $\mathcal{M}^{(0)}_\infty$ has a nice affinoid perfectoid neighbourhood. For that we use the geometry of the Hodge–Tate period morphism. Let $\mathbb{D}^1 \subset \mathbb{P}^1$ be the closed unit disc embedded in $\mathbb{P}^1$ via $x \mapsto [x : 1]$ in usual homogeneous coordinates. Then it is shown in [7] that the preimage $\pi_{\mathrm{HT}}^{-1}(\mathbb{D}^1)$ is affinoid perfectoid and that there exists $0 < \epsilon < 1/2$ such that $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \subset \pi_{\mathrm{HT}}^{-1}(\mathbb{D}^1)$. Fix such an ϵ and define $Y := \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$.

Proposition 3.7. *Let $i \geq 0$ be an integer and let $x \in |\mathcal{M}_\infty^{(0)} \cap Y\gamma^i|$ be a point. Then there exists an open neighbourhood U of x in $\mathcal{M}_\infty^{(0)}$, such that*

- *U is affinoid perfectoid,*
- *$U \subset Y\gamma^i$,*
- *U is invariant under $\gamma^{-i}B(\mathbb{Z}_p)\gamma^i$,*
- *$U \approx \varprojlim U_m$ for affinoid open subsets $U_m \subset \mathcal{M}_m^{(0)}$ and m large enough.*

Proof. We sketch the main ideas of the proof; for details see [4, Prop. 3.7]. For an integer $n \geq 1$, let $z_1, \ldots, z_{p^n} \in \mathbb{Z}_p$ be a set of representatives of $\mathbb{Z}_p/p^n\mathbb{Z}_p$ and consider the rational subset $X_n \subset \mathbb{D}^1$ defined by

$$\begin{aligned} X_n(\mathbb{C}_p) &= \{x \in \mathbb{D}^1(\mathbb{C}_p) : |x - z_i| \geq p^{-n} \text{ for all } i = 1, \ldots, p^n\} \\ &= \{x \in \mathbb{D}^1(\mathbb{C}_p) : |x - z| \geq p^{-n} \text{ for all } z \in \mathbb{Z}_p\}. \end{aligned}$$

It is well known that $(X_n)_{n\in\mathbb{N}}$ is a cover of $\Omega^2 \cap \mathbb{D}^1$. For any $n \geq 0$, the affinoid open $X_n \subset \mathbb{P}^1$ is stable under $B(\mathbb{Z}_p)$. Indeed let $g = \begin{pmatrix} a & b \\ & d \end{pmatrix} \in B(\mathbb{Z}_p)$ and $x \in X_n(\mathbb{C}_p)$, then

$$|x - z| \geq p^{-n}, \ \forall\ z \in \mathbb{Z}_p$$

and therefore

$$\begin{aligned} |x\gamma - z| &= |(dx - b)/a - z| \\ &= |dx - b - az| = |x - (az - b)/d| \\ &\geq p^{-n}. \end{aligned}$$

Therefore $\pi_{\mathrm{HT}}^{-1}(X_n) \subset \pi_{\mathrm{HT}}^{-1}(\mathbb{D}^1)$ is also $B(\mathbb{Z}_p)$-stable. By the results of [7] it is also affinoid perfectoid. Then the intersection

$$U := \pi_{\mathrm{HT}}^{-1}(X_n) \cap Y \cap \mathcal{M}_\infty^{(0)}$$

is still affinoid perfectoid and gives the neighbourhood we want in the case $i = 0$, i.e., $x \in |Y|$.

For $x \in |Y\gamma^i|$, $i \neq 0$, one easily checks that the translates $X_n\gamma^i$ are invariant under the group

$$\gamma^{-i}B(\mathbb{Z}_p)\gamma^i := \left\{ \begin{pmatrix} a & b \\ & d \end{pmatrix} \in B(\mathbb{Q}_p) : a, d \in \mathbb{Z}_p^*, b \in p^{-2i}\mathbb{Z}_p \right\}.$$

The property of being affinoid perfectoid is $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant. One can then check that

$$U := \pi_{\mathrm{HT}}^{-1}(X_n\gamma^i) \cap Y\gamma^i \cap \mathcal{M}_\infty^{(0)}$$

gives an affinoid perfectoid neighbourhood of $x \in Y\gamma^i$, invariant under $\gamma^{-i}B(\mathbb{Z}_p)\gamma^i$. For the "tower property", i.e., the last bullet point we refer to the proof of Proposition 3.7 in [4]. □

Sketch of proof of Theorem 3.5.

Let $x \in |\mathcal{M}_\infty|$ be a point. Assume it is contained in $|Y|$. Choose U as in the proposition. Recall that we have the open map $\phi : Y \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$. Define

$$U/B(\mathbb{Z}_p) := \phi(U) \subset \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a.$$

This is a perfectoid space and indeed the quotient of U by $B(\mathbb{Z}_p)$. If $x \in |Y\gamma^i|$, and $U \approx \varprojlim U_m$ is a good neighbourhood of x as in the last proposition, consider $\phi(U\gamma^{-i})$. We may assume $U\gamma^{-i} \approx \varprojlim W_m$ for $W_m \subset \mathcal{M}_m^{(0)}$ and m large enough. One then shows that $\phi(U\gamma^{-i}) \sim \varprojlim W_m/B(\mathbb{Z}/p^m\mathbb{Z})$. This is not quite the space we want though.

Define

$$H_i := \gamma^i B(\mathbb{Z}_p)\gamma^{-i} = \left\{ \begin{pmatrix} a & b \\ & d \end{pmatrix} \in B(\mathbb{Z}_p) \mid b \in p^{2i}\mathbb{Z}_p \right\} \subset B(\mathbb{Z}_p)$$

and let

$$H_{i,m} \subset B(\mathbb{Z}/p^m\mathbb{Z})$$

be the image of H_i under the natural reduction map $B(\mathbb{Z}_p) \to B(\mathbb{Z}/p^m\mathbb{Z})$. Consider the tower $(W_m/H_{i,m})_{m\geq 2i}$ of affinoid adic spaces

$$W_m/H_{i,m} = \mathrm{Spa}(R_m, R_m^+).$$

Then the natural maps

$$W_m/H_{i,m} \to W_m/B(\mathbb{Z}/p^m\mathbb{Z})$$

are finite étale maps and the degree does not change for m large enough as

$$B(\mathbb{Z}_p)/H_i \to B(\mathbb{Z}/p^m\mathbb{Z})/H_{i,m}$$

is a bijection. In fact the diagram

$$\begin{array}{ccc} W_{m+1}/H_{i,m+1} & \longrightarrow & W_{m+1}/B(\mathbb{Z}/p^{m+1}\mathbb{Z}) \\ \downarrow & & \downarrow \\ W_m/H_{i,m} & \longrightarrow & W_m/B(\mathbb{Z}/p^m\mathbb{Z}) \end{array}$$

is cartesian. The pullback

$$W/H_i := W_{2i}/H_{i,2i} \times_{W_{2i}/B(\mathbb{Z}/p^{2i}\mathbb{Z})} \phi(U\gamma^{-i}) \to \phi(U\gamma^{-i})$$

is finite étale, therefore affinoid perfectoid say $U\gamma^{-i}/H_i = \mathrm{Spa}(R, R^+)$. Define

$$U/B(\mathbb{Z}_p) := \mathrm{Spa}(R, R^+).$$

One now verifies

$$U/B(\mathbb{Z}_p) \approx \varprojlim_m U_m/B(\mathbb{Z}/p^m\mathbb{Z})$$

by checking that the towers $(U_m/B(\mathbb{Z}/p^m\mathbb{Z}))_{m\geq 4i}$ and $(W_m/H_{i,m})_{m\geq 4i}$ are equivalent, i.e., that the systems agree in $(\mathcal{M}_0^{(0)})_{\text{proét}}$.

Finally one glues the spaces $U/B(\mathbb{Z}_p)$ and verifies that the resulting space is indeed the quotient $\mathcal{M}_\infty^{(0)}/B(\mathbb{Z}_p)$ in the category of perfectoid spaces.

□

To get from $\mathcal{M}_\infty^{(0)}/B(\mathbb{Z}_p)$ to a perfectoid space $\mathcal{M}_\infty/B(\mathbb{Q}_p)$ we use the decomposition $\mathcal{M}_\infty = \bigsqcup \mathcal{M}_\infty^{(i)}$ to first reduce to the construction of

$$\mathcal{M}_\infty^{(0)}/B', \quad B' = \{b \in B(\mathbb{Q}_p) | \det(b) \in \mathbb{Z}_p^*\}.$$

This latter space one can construct using the Gross–Hopkins period map at level zero, see [4, Section 3.6].

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