L-INDISTINGUISHABILITY ON EIGENVARIETIES

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Abstract. In this article, we construct examples of L-indistinguishable overconvergent eigenforms for an inner form of SL_2.

1. Introduction

The purpose of this article is to show the existence of points on eigenvarieties for an inner form of SL_2, whose associated systems of Hecke eigenvalues come from classical automorphic representations of an inner form of GL_2, but which are *not classical* themselves, in the sense that there are no classical forms for these systems of Hecke eigenvalues in the corresponding spaces of overconvergent forms. By construction there are *overconvergent* forms giving rise to these points. For each such point we also construct a twin on a different eigenvariety which is classical. The overconvergent non-classical form f on the first eigenvariety and the classical form g on the second eigenvariety are L-indistinguishable in the sense that they give rise to the same Galois representation. Although there is no definition of a global p-adic L-packet, our results suggest that, for any future definition, f and g should lie in the same L-packet.

We describe our results in more detail: Let $B/\mathbb{Q}$ be a definite quaternion algebra and denote by S_B the set of primes where B ramifies. Let $\widetilde{G}$ be the algebraic group over $\mathbb{Q}$ defined by the units B^* and G the subgroup of elements of reduced norm one. Fix a prime $p \notin S_B$ and a finite extension $E/\mathbb{Q}_p$.

For S a finite set of places which includes p and S_B, we have a Hecke algebra $\widetilde{\mathcal{H}}_S := \widetilde{\mathcal{H}}_{ur,S} \otimes_E \widetilde{\mathcal{A}}_p$ for $\widetilde{G}$, which is the product of the spherical Hecke algebras at all places not in S and an Atkin-Lehner algebra at p, and an analogue $\mathcal{H}_S$ for G. Any idempotent $\widetilde{e} = \otimes \widetilde{e}_l \in C_c^\infty(\widetilde{G}(\mathbb{A}_f^p), \overline{\mathbb{Q}})$, such that $\widetilde{e}_l = \mathbf{1}_{\mathrm{GL}_2(\mathbb{Z}_l)}$ for all $l \notin S$, gives rise to an eigenvariety $\mathcal{D}(\widetilde{e})$ of idempotent type $\widetilde{e}$, whose underlying set of points embeds

$$\mathcal{D}(\widetilde{e})(\overline{\mathbb{Q}}_p) \hookrightarrow \mathrm{Hom}(\widetilde{\mathcal{H}}_S, \overline{\mathbb{Q}}_p) \times \widetilde{\mathcal{W}}(\overline{\mathbb{Q}}_p),$$

where $\widetilde{\mathcal{W}} = \mathrm{Hom}_{cts}((\mathbb{Z}_p^*)^2, \mathbb{G}_m)$ denotes the usual weight space. If $e \in C_c^\infty(G(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ is an idempotent with the same set S of bad places, we have an eigenvariety $\mathcal{D}(e)$ of idempotent type e for G, whose underlying set of points embeds

$$\mathcal{D}(e)(\overline{\mathbb{Q}}_p) \hookrightarrow \mathrm{Hom}(\mathcal{H}_S, \overline{\mathbb{Q}}_p) \times \mathcal{W}(\overline{\mathbb{Q}}_p),$$

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for the corresponding weight space $\mathcal{W} = \mathrm{Hom}_{cts}(\mathbb{Z}_p^*, \mathbb{G}_m)$. There are natural maps

$$\mathcal{H}_S \hookrightarrow \widetilde{\mathcal{H}}_S,\ \widetilde{\mathcal{W}} \to \mathcal{W}.$$

Definition (Definition 5.1). *A point z on an eigenvariety of idempotent type is called* classical, *if there is a classical automorphic eigenform in the corresponding space of overconvergent forms, whose system of Hecke eigenvalues is that defined by z.*

Let $\pi(\widetilde{\theta})$ be an algebraic automorphic representation of $\widetilde{G}(\mathbb{A})$ associated to a Größencharacter $\widetilde{\theta}$ of an imaginary quadratic field L. Assume that p splits in L. Then such a representation gives rise to two points on $\mathcal{D}(\widetilde{e})$ for a suitable idempotent $\widetilde{e} \in C_c^\infty(\widetilde{G}(\mathbb{A}_f^p), E)$, one which is ordinary and one which is of critical slope.

Let $\widetilde{x}$ be the point of critical slope and consider its image in $\mathrm{Hom}(\mathcal{H}_S, \overline{\mathbb{Q}}_p) \times \mathcal{W}(\overline{\mathbb{Q}}_p)$ under the composite of the maps

$$\begin{array}{ccc} \mathcal{D}(\widetilde{e})(\overline{\mathbb{Q}}_p) & \longrightarrow & \mathrm{Hom}(\widetilde{\mathcal{H}}_S, \overline{\mathbb{Q}}_p) \times \widetilde{\mathcal{W}}(\overline{\mathbb{Q}}_p) \\ & \overset{\phi}{\searrow} & \downarrow \\ & & \mathrm{Hom}(\mathcal{H}_S, \overline{\mathbb{Q}}_p) \times \mathcal{W}(\overline{\mathbb{Q}}_p), \end{array}$$

which we denote by ϕ.

Our main theorem is the following.

Theorem (Theorem 4.3). *There exist automorphic representations $\pi(\widetilde{\theta})$ of $\widetilde{G}(\mathbb{A})$ as above together with idempotents $\widetilde{e} \in C_c^\infty(\widetilde{G}(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ and $e_1, e_2 \in C_c^\infty(G(\mathbb{A}_f^p), \overline{\mathbb{Q}})$, such that, using the above notation, the image $\phi(\widetilde{x})$ of the critical slope refinement $\widetilde{x}$ of $\pi(\widetilde{\theta})$ lifts to a non-classical point on the eigenvariety $\mathcal{D}(e_1)$ and to a classical point on $\mathcal{D}(e_2)$.*

The proof of the theorem uses a p-adic version of a Labesse–Langlands transfer proved by the author in [10]. The representations $\pi(\widetilde{\theta})$ and the idempotents e_1 and e_2 are constructed in such a way, that each e_i *sees* exactly one member of the L-packet $\Pi(\pi(\widetilde{\theta}))$, i.e., for $i = 1, 2$ there exists a unique element $\pi_i \in \Pi(\pi(\widetilde{\theta}))$ such that $e_i(\pi_i)_f^p \neq 0$. Moreover $m(\pi_1)$, the multiplicity of π_1 in the automorphic spectrum of G, is zero and π_2 is automorphic. In particular, this implies that $\phi(\widetilde{x})$ lifts to $\mathcal{D}(e_2)$. In order to show that it also lifts to $\mathcal{D}(e_1)$, the crucial point is that in a neighbourhood of $\widetilde{x}$ we can find many points associated to automorphic representations that do not come from a Größencharacter. These automorphic representations of $\widetilde{G}(\mathbb{A})$ give rise to stable L-packets of G and therefore their images under ϕ all lift to $\mathcal{D}(e_1)$.

By construction, points on eigenvarieties give rise to systems of Hecke eigenvalues occurring in spaces of overconvergent automorphic forms. So in particular the theorem shows the existence of an overconvergent eigenform f of tame level e_1, whose system of Hecke eigenvalues ψ_f comes from a classical automorphic representation of $\widetilde{G}(\mathbb{A})$. The multiplicity formulae of Labesse and Langlands rule out that there is a classical eigenform for ψ_f of the same tame level. On the other hand the point on $\mathcal{D}(e_2)$ comes from a classical form, say g. The associated Galois representations ρ_f and ρ_g agree as the systems of $\mathcal{H}_S$-eigenvalues for f and g are the same. In this sense the two forms are L-indistinguishable.

Notation. Fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ as well as a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$ and an embedding $E \subset \overline{\mathbb{Q}}_p$. For a number field K we denote by $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of an algebraic closure of K. For a finite set S of places of K, let $G_{K,S}$ be the Galois group of a maximal extension of K that is unramified outside S, and for $v \notin S$, let $\mathrm{Frob}_v \in G_{K,S}$ denote a representative of the geometric Frobenius at v.

We will frequently choose idempotents $e \in C_c^\infty(H(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ where H is equal to either $\widetilde{G}$ or to G. We will always assume that E is chosen big enough so that $\iota_p \circ e$ takes values in E. We ease notation by dropping the embeddings from the notation when it is obvious, e.g., for a complex representation π_f^p of $H(\mathbb{A}_f^p)$ we write $e \cdot \pi_f^p$ instead of $(\iota_\infty \circ e) \cdot \pi_f^p$. Furthermore we assume that the idempotents we consider are given as a tensor product of local idempotents $e_l \in C_c^\infty(H(\mathbb{Q}_l), \overline{\mathbb{Q}})$, where $e_l = \mathbf{1}_{H(\mathbb{Z}_l)}$ for almost all l. We denote by $S(e)$ the minimal finite set of finite primes containing S_B and p, such that $e_l = \mathbf{1}_{H(\mathbb{Z}_l)}$ for all $l \notin S(e)$.

For a rigid analytic space X defined over E, any point is defined over a finite extension of E. We write $X(\overline{\mathbb{Q}}_p) := \bigcup_{E'/E \text{ finite}} X(E')$.

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2. Eigenvarieties and a p-adic Labesse–Langlands transfer

Let $B, \widetilde{G}$ and G be as in Section 1 and let $e := \otimes e_l \in C_c^\infty(G(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ be an idempotent. Choose $S \supset S(e)$ and define

$$\mathcal{H}_{ur,S} := \bigotimes_{l \notin S}{}' \mathcal{H}_E(\mathrm{SL}_2(\mathbb{Q}_l), \mathrm{SL}_2(\mathbb{Z}_l)),$$

where $\mathcal{H}_E(\mathrm{SL}_2(\mathbb{Q}_l), \mathrm{SL}_2(\mathbb{Z}_l))$ is the algebra under convolution of compactly supported E-valued functions on $\mathrm{SL}_2(\mathbb{Q}_l)$ that are bi-invariant under $\mathrm{SL}_2(\mathbb{Z}_l)$. Denote by I the Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ given by

$$I := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) : c \equiv 0 \mod p \right\}.$$

Let

$$\mathcal{H}_S := \mathcal{H}_{ur,S} \otimes_E \mathcal{A}_p$$

be the Hecke algebra, where $\mathcal{A}_p$ is the commutative E-subalgebra of the Iwahori Hecke algebra $\mathcal{H}_E(\mathrm{SL}_2(\mathbb{Q}_p), I)$ generated by the characteristic function on the double coset

$$\mathrm{I}\begin{pmatrix} p^{-1} & \\ & p \end{pmatrix}\mathrm{I}.$$

We let $\mathcal{W} := \mathrm{Hom}_{cts}(\mathbb{Z}_p^*, \mathbb{G}_m)$ be the usual weight space. Using the spaces of overconvergent forms for G as constructed in [9] and Buzzard's machine, one can attach to this data an eigenvariety $\mathcal{D}(e, S)$ of idempotent type e (cf. [4], [9]).

To be precise, one also has to make a choice of a compact operator in the construction, which we fix once and for all to be

$$u_0 := \mathbf{1}_{\mathcal{H}_{ur,S}} \otimes \mathbf{1}_{\mathrm{I}\left(\begin{smallmatrix} p^{-1} & \\ & p \end{smallmatrix}\right)\mathrm{I}} \in \mathcal{H}_S.$$

The eigenvariety $\mathcal{D}(e, S)$ is a rigid analytic space defined over E and it comes equipped with a locally finite (on the source) morphism

$$\omega : \mathcal{D}(e, S) \to \mathcal{W}$$

and an E-algebra homomorphism

$$\psi : \mathcal{H}_S \to \mathcal{O}(\mathcal{D}(e, S)).$$

The points of $\mathcal{D}(e, S)$ correspond to finite slope systems of Hecke eigenvalues occurring in the space of overconvergent forms mentioned above. Moreover the map

$$\begin{array}{rcl} \mathcal{D}(e, S)(\overline{\mathbb{Q}}_p) & \to & \mathrm{Hom}(\mathcal{H}_S, \overline{\mathbb{Q}}_p) \times \mathcal{W}(\overline{\mathbb{Q}}_p) \\ x & \mapsto & (\psi_x(h) := \psi(h)(x), \omega(x)) \end{array}$$

is an injection (cf. Lemma 7.2.7 of [1]).

Likewise, starting from an idempotent $\widetilde{e} \in C_c^\infty(\widetilde{G}(\mathbb{A}_f^p), \overline{\mathbb{Q}})$, a set $S \supset S(\widetilde{e})$ and an associated Hecke algebra $\widetilde{\mathcal{H}}_S = \widetilde{\mathcal{H}}_{ur,S} \otimes_E \widetilde{\mathcal{A}}_p$, we can build an eigenvariety $\mathcal{D}(\widetilde{e}, S)$ of idempotent type $\widetilde{e}$ for $\widetilde{G}$. The weight space in this case is $\widetilde{\mathcal{W}} := \mathrm{Hom}_{cts}((\mathbb{Z}_p^*)^2, \mathbb{G}_m)$. We have natural maps (see [10, Sections 2.2 and 2.3])

$$\mu : \widetilde{\mathcal{W}} \longrightarrow \mathcal{W},$$

$$\lambda : \mathcal{H}_S \hookrightarrow \widetilde{\mathcal{H}}_S.$$

In the construction of $\mathcal{D}(\widetilde{e}, S)$ we always choose $\lambda(u_0)$ as the compact operator.

There are natural $\mathbb{Q}$-structures on the Hecke algebras defined by the subalgebras of $\mathbb{Q}$-valued functions which we denote by $\mathcal{H}_{\mathbb{Q},S}, \mathcal{H}_{\mathbb{Q},ur,S}$ etc. The monomorphism λ can in fact be defined over $\mathbb{Q}$ (see Section 2 of [10] for details). In particular we constructed in Lemma 2.10 of [10] an inclusion of $\mathbb{Q}$-algebras

$$\lambda_{l,\mathbb{Q}} : \mathcal{H}_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q}_l), \mathrm{SL}_2(\mathbb{Z}_l)) \hookrightarrow \mathcal{H}_{\mathbb{Q}}(\mathrm{GL}_2(\mathbb{Q}_l), \mathrm{GL}_2(\mathbb{Z}_l)).$$

We also refer to [10] for details regarding the construction as well as general properties of the eigenvarieties.

Remark 2.1. *There exists a Zariski-dense and accumulation subset*

$$Z \subset \mathcal{D}(e, S)(\overline{\mathbb{Q}}_p)$$

coming from p-refined classical automorphic representations as defined in [10]*, Def. 3.14. This is proved as usual (cf.* [5, Section 6.4.5] *and also* [10, Prop. 3.9] *for* *a proof of the analogous assertion for $\widetilde{G}$ in the same notation) using the fact that forms of small slope are classical and that classical weights are dense in weight space.*

Define $t_l \in \mathcal{H}_{ur,S}$, as the characteristic function on the double coset

$$\mathrm{SL}_2(\widehat{\mathbb{Z}}^S)\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}\mathrm{SL}_2(\widehat{\mathbb{Z}}^S),$$

where $\left(\begin{smallmatrix} l & \\ & l^{-1} \end{smallmatrix}\right)$ is understood to be the matrix in $\mathrm{SL}_2(\widehat{\mathbb{Z}}^S) = \prod_{q \notin S} \mathrm{SL}_2(\mathbb{Z}_q)$ which is equal to 1 for all $q \neq l$ and equal to $\left(\begin{smallmatrix} l & \\ & l^{-1} \end{smallmatrix}\right)$ at l. Furthermore let

$$h_l := \frac{1}{l}(t_l + 1) \in \mathcal{H}_{ur,S}.$$

Note that h_l is an element of the subalgebra $\mathcal{H}^0_{ur,S} \subset \mathcal{H}_{ur,S}$ of $\mathcal{O}_E$-valued functions.

Lemma 2.2. *Let $\mathcal{D}(e, S)$ be an eigenvariety of idempotent type for G. Then there exists a 3-dimensional pseudo-representation*

$$T : G_{\mathbb{Q},S} \to \mathcal{O}(\mathcal{D}(e, S))$$

such that $T(\mathrm{Frob}_l) = \psi(h_l)$ for all $l \notin S$.

Proof. By the previous remark we have a Zariski dense subset $Z \subset \mathcal{D}(e, S)(\overline{\mathbb{Q}}_p)$ of classical points. A point $z \in Z$ comes from an algebraic automorphic representation π of $G(\mathbb{A})$ and there is a projective Galois representation

$$\rho_z : G_{\mathbb{Q},S} \to \mathrm{PGL}_2(\overline{\mathbb{Q}}_p)$$

associated to π. Namely, if $\widetilde{\pi}$ is any algebraic automorphic representation of $\widetilde{G}(\mathbb{A})$ which is unramified outside S and lifts π, let $\rho(\widetilde{\pi}) : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the attached Galois representation by Deligne. It has the property that for any $l \notin S$, the characteristic polynomial of $\rho(\widetilde{\pi})(\mathrm{Frob}_l)$ is given by $X^2 - T_l(\widetilde{\pi})X + lS_l(\widetilde{\pi})$, where $T_l(\widetilde{\pi}) := \iota_p \circ \iota_\infty^{-1}(\mu_l)$ and μ_l is the eigenvalue of

$$T_l := \mathbf{1}_{\mathrm{GL}_2(\widehat{\mathbb{Z}}^S)\left(\begin{smallmatrix} l & \\ & 1 \end{smallmatrix}\right)\mathrm{GL}_2(\widehat{\mathbb{Z}}^S)} \in \widetilde{\mathcal{H}}_{\mathbb{Q},ur,S}$$

on $(\widetilde{\pi}_f^S)^{\mathrm{GL}_2(\widehat{\mathbb{Z}}^S)}$ and similarly for $S_l = \mathbf{1}_{\mathrm{GL}_2(\widehat{\mathbb{Z}}^S)\left(\begin{smallmatrix} l & \\ & l \end{smallmatrix}\right)\mathrm{GL}_2(\widehat{\mathbb{Z}}^S)} \in \widetilde{\mathcal{H}}_{\mathbb{Q},ur,S}$.

Then

$$\rho_z = \eta \circ \rho(\widetilde{\pi}) : G_{\mathbb{Q},S} \to \mathrm{PGL}_2(\overline{\mathbb{Q}}_p)$$

is the composition of $\rho(\widetilde{\pi})$ and the natural homomorphism $\eta : \mathrm{GL}_2(\overline{\mathbb{Q}}_p) \to \mathrm{PGL}_2(\overline{\mathbb{Q}}_p)$. There is a monomorphism $\iota : \mathrm{PGL}_2(\overline{\mathbb{Q}}_p) \hookrightarrow \mathrm{GL}_3(\overline{\mathbb{Q}}_p)$ coming from the adjoint action, which identifies $\mathrm{PGL}_2(\overline{\mathbb{Q}}_p)$ with $\mathrm{SO}_3(\overline{\mathbb{Q}}_p)$. Define

$$\sigma_z := \iota \circ \rho_z : G_{\mathbb{Q},S} \to \mathrm{GL}_3(\overline{\mathbb{Q}}_p).$$

Then

$$\sigma_z \cong \mathrm{Sym}^2(\rho(\widetilde{\pi})) \otimes \det(\rho(\widetilde{\pi}))^{-1}$$

and an easy calculation shows that for all $l \notin S$

$$\begin{aligned}\mathrm{Tr}(\sigma_z(\mathrm{Frob}_l)) &= \mathrm{Tr}^2(\rho(\widetilde{\pi})(\mathrm{Frob}_l))/\det(\rho(\widetilde{\pi})(\mathrm{Frob}_l)) - 1 \\ &= T_l^2(\widetilde{\pi})/(lS_l(\widetilde{\pi})) - 1.\end{aligned}$$

Now an elementary calculation shows that $T_l^2/(lS_l) - 1 = \lambda_{\mathbb{Q}}(h_l)$ and therefore

$$T_l^2(\widetilde{\pi})/(lS_l(\widetilde{\pi})) - 1 = \psi(h_l)(z).$$

The lemma now follows from Proposition 7.1.1 of [5] (see also Section 3.1.3 of [10]), i.e., Hypothesis **H** in [5] is satisfied using the Zariski-dense set $Z \subset \mathcal{D}(e,S)(\overline{\mathbb{Q}}_p)$, the representations σ_z for $z \in Z$ and the family of functions $\psi(h_l) \in \mathcal{O}(\mathcal{D}(e,S))$. □

Remark 2.3. *In the following we abbreviate* $\mathcal{D}(e) := \mathcal{D}(e, S(e))$ *and* $\mathcal{D}(\widetilde{e}) := \mathcal{D}(\widetilde{e}, S(\widetilde{e}))$*.*

Below we will often use so-called *special idempotents* attached to a finite set of Bernstein components. Let $F/\mathbb{Q}_l$ be a finite extension, $H(F)$ the F-points of a reductive group over F. Given a Bernstein component $\mathfrak{s}$ of the category of smooth $\overline{\mathbb{Q}}$-representations of $H(F)$, there is an idempotent $e_{\mathfrak{s}} \in C_c^\infty(H(F), \overline{\mathbb{Q}})$ such that for an irreducible smooth $\overline{\mathbb{Q}}$-representation σ of $H(F)$, $e_{\mathfrak{s}} \cdot \sigma \neq 0$ if and only if σ is contained in the Bernstein component $\mathfrak{s}$. Similarly one can attach an idempotent to a finite set Σ of Bernstein components. We refer to Section 3 of [3] for a nice overview and to Proposition 3.13 of loc.cit. for the existence of these so-called special idempotents.

When we choose special idempotents below, we may always assume they take values in $\overline{\mathbb{Q}}$ as all automorphic representations we deal with in this paper are algebraic and have the property that their finite part is defined over $\overline{\mathbb{Q}}$.

We also want to remark here that irreducible supercuspidal representations that are in the same Bernstein component differ from each other by a twist by an unramified character. In particular, if two irreducible supercuspidal representations σ and σ' of $\mathrm{SL}_2(\mathbb{Q}_l)$ are in the same Bernstein component, then $\sigma \cong \sigma'$.

We use the following notation: Let $\mathfrak{s}$ be the Bernstein component of $\widetilde{G}(\mathbb{Q}_l)$ defined by a supercuspidal representation $\widetilde{\pi}_l$ of $\widetilde{G}(\mathbb{Q}_l)$. Then we denote by $\mathrm{Res}_G^{\widetilde{G}}(\mathfrak{s})$ the finite set of Bernstein components defined by the representations occurring in $\widetilde{\pi}_l|_{G(\mathbb{Q}_l)}$. Note this is well defined.

The classical transfer, by which we just mean the map that attaches to an automorphic representation $\widetilde{\pi}$ of $\widetilde{G}(\mathbb{A})$ an L-packet of representations $\Pi(\widetilde{\pi})$ of $G(\mathbb{A})$, can be interpolated to maps between suitable eigenvarieties.

Definition 2.4. *Two idempotents* $\widetilde{e} \in C_c^\infty(\widetilde{G}(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ *and* $e \in C_c^\infty(G(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ *are called* Langlands compatible *if they satisfy: For any discrete automorphic representation* $\widetilde{\pi}$ *of* $\widetilde{G}(\mathbb{A})$ *with* $\widetilde{e} \cdot \widetilde{\pi}_f^p \neq 0$ *and any* $\tau \in \Pi(\widetilde{\pi}_p)$*, there exists an element* π *in the packet* $\Pi(\widetilde{\pi})$*, such that*

- $m(\pi) > 0$,
- $e \cdot \pi_f^p \neq 0$ *and*
- $\pi_p = \tau$.

Theorem 2.5. *(1) Let* $\widetilde{e} \in C_c^\infty(\widetilde{G}(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ *be an idempotent. Then there exists an idempotent* $e \in C_c^\infty(G(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ *with* $S(e) = S(\widetilde{e})$ *and such that* $\widetilde{e}$ *and* e *are Langlands compatible. Define S:=S(e).*

(2) *Assume* $\widetilde{e} = \otimes \widetilde{e}_l \in C_c^\infty(\widetilde{G}(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ *has the property that for all* $l \in S(\widetilde{e})$, $\widetilde{e}_l$ *is a special idempotent attached to a supercuspidal Bernstein component* $\mathfrak{s}_l$. *Define* $e := \otimes e_l \in C_c^\infty(G(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ *where* $e_l := e_{\mathrm{SL}_2(\mathbb{Z}_l)}$ *for all* $l \notin S(\widetilde{e})$ *and for* $l \in S(\widetilde{e})$, e_l *is a special idempotent attached to the finite set of Bernstein components* $\mathrm{Res}_G^{\widetilde{G}}(\mathfrak{s}_l)$. *Then* $\widetilde{e}$ *and* e *are Langlands compatible.*

(3) *For any two Langlands compatible idempotents* $\widetilde{e} \in C_c^\infty(\widetilde{G}(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ *and* $e \in C_c^\infty(G(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ *with the same set* S *of bad places there exists a morphism* $\zeta : \mathcal{D}(\widetilde{e}) \to \mathcal{D}(e)$ *such that the diagrams*

$$\begin{array}{ccc} \mathcal{D}(\widetilde{e}) & \xrightarrow{\ \zeta\ } & \mathcal{D}(e) \\ \downarrow{\scriptstyle \widetilde{\omega}} & & \downarrow{\scriptstyle \omega} \\ \widetilde{\mathcal{W}} & \xrightarrow{\ \mu\ } & \mathcal{W} \end{array} \qquad \begin{array}{ccc} \mathcal{H}_S & \xhookrightarrow{\ \lambda\ } & \widetilde{\mathcal{H}}_S \\ \downarrow & & \downarrow \\ \mathcal{O}(\mathcal{D}(e)) & \xrightarrow{\ \zeta^*\ } & \mathcal{O}(\mathcal{D}(\widetilde{e})) \end{array}$$

commute.

Proof. Part (1) follows from Proposition 4.15 and Proposition 4.16 of [10], once we remark that for any idempotent $\widetilde{e} \in C_c^\infty(\widetilde{G}(\mathbb{A}_f^p), \overline{\mathbb{Q}})$, there exists a compact open subgroup $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_f^p)$ such that $e_{\widetilde{K}} \cdot \widetilde{e} = \widetilde{e} = \widetilde{e} \cdot e_{\widetilde{K}}$. Part (3) follows from [10] Theorem 5.7 and the proof of it.

Part (2) can be proved in the same way as Proposition 4.16 of [10]. Namely for an automorphic representation $\widetilde{\pi}$ of $\widetilde{G}(\mathbb{A})$ with $\widetilde{e} \cdot \widetilde{\pi}_f^p \neq 0$ and any $\tau \in \Pi(\widetilde{\pi}_p)$ define

$$\begin{aligned} Y(\widetilde{\pi}, \tau) &:= \{\pi \in \Pi(\widetilde{\pi}) | e \cdot (\pi_f^p) \neq 0, \pi_p = \tau\} \\ &= \{\pi \in \Pi(\widetilde{\pi}) | \pi_l = \pi_l^0 \ \forall l \notin S(\widetilde{e}), \pi_p = \tau\}, \end{aligned}$$

where in the last line π_l^0 denotes the unique member of the L-packet $\Pi(\widetilde{\pi}_l)$ with $(\pi_l^0)^{\mathrm{SL}_2(\mathbb{Z}_l)} \neq 0$. We need to show that there exists $\pi \in Y(\widetilde{\pi}, \tau)$ such that $m(\pi) > 0$. For that let $\pi \in Y(\widetilde{\pi}, \tau)$ be arbitrary and assume $m(\pi) = 0$. Then by Proposition 4.11 of [10] we may change π at a prime $l \in S_B$ to a different representation in the local L-packet $\Pi(\widetilde{\pi}_l)$ to get a representation π' which is automorphic and still in $Y(\widetilde{\pi}, \tau)$. □

Remark 2.6. *We recall that* ζ *is constructed using two auxiliary eigenvarieties* $\mathcal{D}'(\widetilde{e})$ *and* $\mathcal{D}''(e)$, *which are described in Section 3.3 and 3.4 of* [10]. $\mathcal{D}'(\widetilde{e})$ *is the eigenvariety which apart from the Hecke algebra is build from the same data as* $\mathcal{D}(\widetilde{e})$ *but where the Hecke algebra is replaced by* $\mathcal{H}_S$. *It comes equipped with a morphism* $\omega' : \mathcal{D}'(\widetilde{e}) \to \widetilde{\mathcal{W}}$ *and the points of* $\mathcal{D}'(\widetilde{e})$ *embed*

$$\mathcal{D}'(\widetilde{e})(\overline{\mathbb{Q}}_p) \hookrightarrow \mathrm{Hom}(\mathcal{H}_S, \overline{\mathbb{Q}}_p) \times \widetilde{\mathcal{W}}(\overline{\mathbb{Q}}_p).$$

We have a morphism $\lambda' : \mathcal{D}(\widetilde{e}) \to \mathcal{D}'(\widetilde{e})$, *which on points is given by*

$$(\psi_x, \widetilde{\omega}(x)) \mapsto (\psi_x|_{\mathcal{H}_S}, \widetilde{\omega}(x)).$$

The second eigenvariety $\mathcal{D}''(e)$ is simply defined as the pullback $\widetilde{\mathcal{W}} \times_{\mathcal{W}} \mathcal{D}(e)$ and the morphism ζ is the composite

$$\begin{array}{ccccccc} \mathcal{D}(\widetilde{e}) & \xrightarrow{\lambda'} & \mathcal{D}'(\widetilde{e}) & \xrightarrow{\xi} & \mathcal{D}''(e) & \longrightarrow & \mathcal{D}(e) \\ \downarrow{\scriptstyle \widetilde{\omega}} & & \downarrow{\scriptstyle \omega'} & & \downarrow & & \downarrow{\scriptstyle \omega} \\ \widetilde{\mathcal{W}} & \xrightarrow{\text{id}} & \widetilde{\mathcal{W}} & \xrightarrow{\text{id}} & \widetilde{\mathcal{W}} & \xrightarrow{\mu} & \mathcal{W} \end{array} .$$

By Proposition 5.6 of [10], *the morphism ξ is a closed immersion. This is important in what follows.*

Remark 2.7. *If $e, e' \in C_c^\infty(G(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ are two idempotents as above, such that $S(e) = S(e')$ and such that $e_l'|e_l$ for all $l \in S\backslash\{p\}$, then there exists a closed immersion $\mathcal{D}(e) \hookrightarrow \mathcal{D}(e')$. (cf.* [1] *Section 7.3).*

3. Slopes of CM points

We determine the slopes of points on eigenvarieties $\mathcal{D}(\widetilde{e})$ that arise from automorphic representations $\pi(\widetilde{\theta})$ of $\widetilde{G}(\mathbb{A})$ coming from a Größencharacter.

We view $\mathbb{Z}^2$ as a subset of weight space via

$$\mathbb{Z}^2 \hookrightarrow \widetilde{\mathcal{W}}(\overline{\mathbb{Q}}_p), (k_1, k_2) \mapsto \Big((z_1, z_2) \mapsto z_1^{k_1} z_2^{k_2}\Big).$$

Let $\underline{k} = (k_1, k_2) \in \mathbb{Z}^2$, $k_1 \geq k_2$. We denote by $\widetilde{I}$ the Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ given by

$$\widetilde{I} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : c \equiv 0 \mod p \right\}.$$

Recall (cf. Section 7.2.2 of [1] and Definition 3.14 of [10]) that a p-refined automorphic representation of weight $\underline{k}$ of $\widetilde{G}(\mathbb{A})$ is a pair $(\widetilde{\pi}, \chi)$ such that

- $\widetilde{\pi}$ is an automorphic representation of $\widetilde{G}(\mathbb{A})$;
- $\widetilde{\pi}_p$ has a non-zero fixed vector under the Iwahori $\widetilde{I}$ and $\chi = (\chi_1, \chi_2)$ is an ordered pair of characters $\chi_i : \overline{\mathbb{Q}}_p^* \to \mathbb{C}^*, i = 1, 2$ such that $\widetilde{\pi}_p \hookrightarrow \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1, \chi_2)$, where $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(-)$ denotes the normalized parabolic induction from the upper triangular Borel $B \subset \mathrm{GL}_2(\mathbb{Q}_p)$;
- $\widetilde{\pi}_\infty \cong (\mathrm{Sym}^{k_1 - k_2}(\mathbb{C}^2) \otimes \mathrm{Nrd}^{k_2})^*$.

Different points on an eigenvariety that come from the same automorphic representation $\widetilde{\pi}$ are parametrized by the different choices of pairs $\chi = (\chi_1, \chi_2)$, such that $\widetilde{\pi_p} \hookrightarrow \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1, \chi_2)$, which are called refinements. Note if $\pi_p \cong \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1, \chi_2)$, there are precisely two refinements, namely (χ_1, χ_2) and (χ_2, χ_1).

Fix an idempotent $\widetilde{e}$ and let $S := S(\widetilde{e})$. Define

$$U_p := \mathbf{1}_{\widetilde{\mathcal{H}}_{ur,S}} \otimes \mathbf{1}_{\widetilde{\mathrm{I}} \begin{pmatrix} 1 & \\ & p \end{pmatrix} \widetilde{\mathrm{I}}} \in \widetilde{\mathcal{H}}_S.$$

The operator

$$\mathbf{1}_{\widetilde{\mathcal{H}}_{ur,S}} \otimes \mathbf{1}_{\widetilde{\mathrm{I}} \begin{pmatrix} p^{-1} & \\ & p \end{pmatrix} \widetilde{\mathrm{I}}} \in \widetilde{\mathcal{H}}_S.$$

is the image of u_0 under λ, and we denote it by u_0 again.

Recall from Section 2, that $\mathcal{D}(\widetilde{e})$ comes equipped with a morphism $\psi : \widetilde{\mathcal{H}}_S \to \mathcal{O}(\mathcal{D}(\widetilde{e}))$. For a point z on the eigenvariety $\mathcal{D}(\widetilde{e})$ which corresponds to a p-refined automorphic representation $(\widetilde{\pi}, (\chi_1, \chi_2))$ of weight $\underline{k}$, we have

$$\psi(U_p)(z) = \psi_{(\widetilde{\pi},(\chi_1,\chi_2))}(U_p) = \iota_p(\chi_2(p)p^{1/2})p^{-k_2}$$

and

$$\psi(u_0)(z) = \iota_p(\chi_2(p)\chi_1(p)^{-1})p^{k_1-k_2+1}. \tag{1}$$

To justify the next definition recall the following classicality theorem. Again fix $\underline{k}$ as above and choose an affinoid neighbourhood X of $\underline{k}$ in $\widetilde{\mathcal{W}}$. We denote by $M(\widetilde{e}, \underline{k}, k(X))$ the E-Banach space of overconvergent forms of weight $\underline{k}$ and 'tame level' $\widetilde{e}$ as defined in Section 3 of [9]. It comes equipped with an action of $\widetilde{\mathcal{H}}_S$. We sometimes omit the parameter $k(X)$ and write $M(\widetilde{e}, \underline{k})$ instead. If $x \in \mathcal{D}(\widetilde{e})(\overline{\mathbb{Q}}_p)$ with $\widetilde{\omega}(x) = \underline{k}$, then there exists an overconvergent finite slope eigenform $f \in M(\widetilde{e}, \underline{k}, k(X))$ with eigenvalues ψ_x. The space $M(\widetilde{e}, \underline{k}, k(X))$ has a finite dimensional subspace $M(\widetilde{e}, \underline{k})^{cl}$ of classical forms.

Theorem 3.1 ([9, Theorem 3.9.6]). *Let $\underline{k} = (k_1, k_2) \in \mathbb{Z}^2$, $k_1 \geq k_2$. Let E'/E be a finite extension, $\lambda \in E'^*$ and $\sigma := v_p(\lambda)$. If*

$$\sigma < k_1 - k_2 + 1,$$

then the generalized λ-eigenspace of U_p acting on $M(\widetilde{e}, \underline{k}, k(X))\widehat{\otimes}_E E'$ is contained in the subspace $M(\widetilde{e}, \underline{k})^{cl}\widehat{\otimes}_E E'$.

Definition 3.2. *(1) A point $x = (\psi_x, \widetilde{\omega}(x))$ on $\mathcal{D}(\widetilde{e})$, with $\widetilde{\omega}(x) = (k_1, k_2)$ is called* of critical slope *if $v_p(\psi_x(U_p)) = k_1 - k_2 + 1$.*

(2) A refinement χ of an automorphic representation $\widetilde{\pi}$ of $\widetilde{G}(\mathbb{A})$ of weight (k_1, k_2) is called of critical slope *if $v_p(\psi_{(\widetilde{\pi},\chi)}(U_p)) = k_1 - k_2 + 1$.*

Now let $L/\mathbb{Q}$ be an imaginary quadratic extension and let $\widetilde{\theta} : \mathbb{A}_L^*/L^* \to \mathbb{C}^*$ be a Größencharacter which does not factor through the norm. In [7], Jacquet and Langlands show how to associate to $\widetilde{\theta}$ a cuspidal automorphic representation $\tau(\widetilde{\theta})$ of $\mathrm{GL}_2(\mathbb{A})$. We refer to §12 of [7] for details regarding the construction and characterization. Assume $\tau(\widetilde{\theta})$ is in the image of the global Jacquet–Langlands transfer JL from $\widetilde{G}$ to GL_2, i.e., $\tau(\widetilde{\theta})_v$ is a discrete series representation for all $v \in S_B$. Then $\pi(\widetilde{\theta}) := \mathrm{JL}^{-1}(\tau(\widetilde{\theta}))$ is an automorphic representation of $\widetilde{G}(\mathbb{A})$. Assume $\pi(\widetilde{\theta})$ is of weight $(k_1, k_2) \in \mathbb{Z}^2$, so

$$\pi(\widetilde{\theta})_\infty \cong (\mathrm{Sym}^{k_1-k_2}(\mathbb{C}^2) \otimes \mathrm{Nrd}^{k_2})^* \cong \mathrm{Sym}^{k_1-k_2}(\mathbb{C}^2) \otimes \mathrm{Nrd}^{-k_1}$$

and $\widetilde{\theta}_\infty : L_\infty^* \to \mathbb{C}^*$ is given by $\widetilde{\theta}_\infty(z) = (z\overline{z})^{-k_1-1/2} z^{k_1-k_2+1}$ (see Remark 7.7 of [6]). Let $\widetilde{e} \in C_c^\infty(\widetilde{G}(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ be an idempotent such that $\widetilde{e} \cdot \pi(\widetilde{\theta})_f^p \neq 0$.

Lemma 3.3. *Let $\pi(\widetilde{\theta})$ be an automorphic representation of $\widetilde{G}(\mathbb{A})$ associated to a Größencharacter of L as above. Assume p splits in L and $\pi(\widetilde{\theta})_p$ is unramified. Then $\pi(\widetilde{\theta})$ has a refinement of critical slope. More precisely let $\widetilde{x}, \widetilde{y} \in \mathcal{D}(\widetilde{e})$ be the* *two points attached to* $\pi(\widetilde{\theta})$*. Then the slopes of* $\psi_{\widetilde{x}}(U_p)$ *and* $\psi_{\widetilde{y}}(U_p)$ *are* $k_1 - k_2 + 1$ *and* 0*. Furthermore*

$$v_p(\psi_{\widetilde{x}}(u_0)) = 2(k_1 - k_2 + 1).$$

Proof. The Größencharacter $\widetilde{\theta}_0 := \widetilde{\theta}|| \mathrm{N}_{L/\mathbb{Q}}(\cdot)||^{-1/2}$ is algebraic and we can turn it into a p-adic character by shifting the weight from ∞ to p, i.e., we define

$$\widetilde{\theta}' : \mathbb{A}_L^*/L^* \to \overline{\mathbb{Q}}_p^*$$

$$\widetilde{\theta}'(x) = \iota_p(\widetilde{\theta}_0(x)\widetilde{\theta}_{0,\infty}^{-1}(x_\infty))\tau_w(x_w)^{-k_2}\tau_{\overline{w}}(x_{\overline{w}})^{-k_1-1},$$

where we have matched the two complex embeddings of L_∞ with the two places $w, \overline{w}$ above p. The finite part of an algebraic Größencharacter takes values in a number field. Moreover, $\widetilde{\theta}'$ factors through the compact group $\mathbb{A}_L^*/L^*L_\infty^*$, so it takes values in $\mathcal{O}_F^*$ for some finite extension $F/\mathbb{Q}_p$.

Our assumptions imply that $\pi(\widetilde{\theta})_p \cong \mathrm{Ind}_B^{\mathrm{GL}_2(\overline{\mathbb{Q}}_p)}(\widetilde{\theta}_w, \widetilde{\theta}_{\overline{w}})$ and the two refinements of $\pi(\widetilde{\theta})$ are given by $(\widetilde{\theta}_w, \widetilde{\theta}_{\overline{w}})$ and $(\widetilde{\theta}_{\overline{w}}, \widetilde{\theta}_w)$.

Let $p_w = (1, \ldots, 1, p, 1, \ldots, 1) \in \mathbb{A}_L^*$ (respectively $p_{\overline{w}}$) denote the idele which is 1 at all places except for w (respectively $\overline{w}$), where it equals p. Then

$$\widetilde{\theta}'(p_w) = \iota_p(\widetilde{\theta}_w(p)p^{1/2})p^{-k_2} = \psi_{(\pi(\widetilde{\theta}),(\widetilde{\theta}_{\overline{w}},\widetilde{\theta}_w))}(U_p) \quad \text{and}$$

$$\widetilde{\theta}'(p_{\overline{w}}) = \iota_p(\widetilde{\theta}_{\overline{w}}(p)p^{1/2})p^{-k_1-1} = \psi_{(\pi(\widetilde{\theta}),(\widetilde{\theta}_w,\widetilde{\theta}_{\overline{w}}))}(U_p)p^{k_2-k_1-1}.$$

As $\widetilde{\theta}'(p_w)$ and $\widetilde{\theta}'(p_{\overline{w}})$ are in $\mathcal{O}_F^*$ this implies the claim on the slopes of the U_p-eigenvalues. Using Equation (1), one then verifies the slope of $\psi_{\widetilde{x}}(u_0)$. ☐

Remark 3.4. *The character* $\widetilde{\theta}' : \mathbb{A}_L^*/L^* \to \overline{\mathbb{Q}}_p^*$ *in the proof of the above lemma is trivial on* L_∞^**, so it factors through the quotient* $\mathbb{A}_L^*/L^*L_\infty^* \cong G_L^{ab}$*. We may therefore view* $\widetilde{\theta}'$ *as a continuous character of* G_L *with values in* $\overline{\mathbb{Q}}_p^*$*. In this notation the Galois representation* $\rho_{\pi(\widetilde{\theta})} : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ *attached to* $\pi(\widetilde{\theta})$ *is given by*

$$\rho_{\pi(\widetilde{\theta})} \cong \mathrm{Ind}_{G_L}^{G_\mathbb{Q}}(\widetilde{\theta}')$$

as one easily checks by comparing traces of Hecke operators and Frobenius.

Lemma 3.5. *Let* $\pi(\widetilde{\theta})$ *be an automorphic representation of* $\widetilde{G}(\mathbb{A})$ *of tame level* $\widetilde{e}$*, weight* (k_1, k_2) *and unramified at* p *which is associated to a Größencharacter* $\widetilde{\theta} : \mathbb{A}_L^*/L^* \to \mathbb{C}^*$ *and assume that* p *is inert in* L*. Then* $\pi(\widetilde{\theta})$ *gives rise to two distinct points* x, y *on* $\mathcal{D}(\widetilde{e})$*. Their slopes agree and are equal to*

$$v_p(\psi_x(U_p)) = v_p(\psi_y(U_p)) = (k_1 - k_2 + 1)/2.$$

Proof. Let v denote the unique place above p, and let $\omega : \mathbb{Q}_p^* \to \mathbb{C}^*$ be the character associated to the quadratic extension $L_v/\mathbb{Q}_p$ by local class field theory. By assumption $\widetilde{\theta}_v$ is unramified and therefore factors through the norm $\mathrm{N}_{L_v/\mathbb{Q}_p} : L_v^* \to \mathbb{Q}_p^*$.

Let $\delta : \mathbb{Q}_p^* \to \mathbb{C}^*$ be a character such that $\widetilde{\theta}_v = \delta \circ \mathrm{N}_{L_v/\mathbb{Q}_p}$. Then by construction $\pi(\widetilde{\theta})_p = \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\delta, \delta\omega)$, in particular, the two refinements are distinct. The U_p-eigenvalues on the two points x and y are given by

$$\psi_x(U_p) = \iota_p(\delta(p)p^{1/2})p^{-k_2},$$

$$\psi_y(U_p) = \iota_p(\delta(p)\omega(p)p^{1/2})p^{-k_2} = \iota_p(-\delta(p)p^{1/2})p^{-k_2}.$$

In particular, we see that they have the same slope. A similar calculation as in the proof of the last lemma, with

$$\widetilde{\theta}'(x) = \iota_p(\widetilde{\theta}_0(x)\widetilde{\theta}_{0,\infty}^{-1}(x_\infty))(\mathrm{N}_{L_v/\mathbb{Q}_p}(x_v))^{-k_1-1}\tau_v(x_v)^{k_1-k_2+1},$$

implies that the slope is given by $(k_1 - k_2 + 1)/2$. □

4. Existence of L-indistinguishable forms

Let $q \geq 5$ be a prime number such that $-q \equiv 1 \mod 4$ and let $L := \mathbb{Q}(\sqrt{-q})$ be the associated imaginary quadratic extension. Choose a prime p which splits in L and let B be the quaternion algebra over $\mathbb{Q}$ such that $S_B = \{q, \infty\}$. Let $\widetilde{G}$ and G be as above.

Let $\pi(\widetilde{\theta})$ be an automorphic representation of $\widetilde{G}(\mathbb{A})$ coming from a Größencharacter $\widetilde{\theta} : \mathbb{A}_L^*/L^* \to \mathbb{C}^*$ of L. Assume that

(1) $\pi(\widetilde{\theta})_l$ is unramified for all $l \neq q$.
(2) The L-packet $\Pi(\pi(\widetilde{\theta})_q) = \{\tau_1, \tau_2\}$ defined by $\pi(\widetilde{\theta})_q$ is of size two.
(3) Precisely one of the representations

$$\pi_1 := \bigotimes_{l\neq q} \pi_l^0 \otimes \tau_1 \otimes \pi_\infty, \ \pi_2 := \bigotimes_{l\neq q} \pi_l^0 \otimes \tau_2 \otimes \pi_\infty$$

is automorphic. As before π_l^0 denotes the unique member of the local L-packet $\Pi(\pi(\widetilde{\theta})_l)$, which has a non-zero fixed vector under $\mathrm{SL}_2(\mathbb{Z}_l)$.

Lemma 4.1. *Automorphic representations $\pi(\widetilde{\theta})$ of $\widetilde{G}(\mathbb{A})$ satisfying the above list of properties exist.*

Proof. Note that $(\prod_{w\neq\infty} \mathcal{O}_w^* \times L_\infty^*)/\mathcal{O}_L^* \hookrightarrow \mathbb{A}_L^*/L^*$ is of finite index. Furthermore, our assumptions imply that $\mathcal{O}_L^* = \{1, -1\}$ and that q is the only prime that ramifies in L. Define a Größencharacter $\widetilde{\theta} : \mathbb{A}_L^*/L^* \to \mathbb{C}^*$ as follows:

Let $\widetilde{\theta}_\infty : L_\infty^* \to \mathbb{C}^*$ be the character given by $\widetilde{\theta}_\infty(z) \mapsto (z\overline{z})^r z^m$, where $r \in \mathbb{C}$ and $m \geq 2$ is an even integer, so that $\widetilde{\theta}_\infty$ is trivial on $\mathcal{O}_L^*$.

Denote by v the unique place of L above q. For all $w \neq v$ let $\widetilde{\theta}_w : \mathcal{O}_w^* \to \mathbb{C}^*$ be the trivial character.

For v, let $\mathcal{O}_v^1$ be the kernel of the norm map $(\mathrm{N}_{L_v/\mathbb{Q}_q})|_{\mathcal{O}_v^*}$. Choose any continuous non-quadratic character

$$\theta_v' : \mathcal{O}_v^1/\{1, -1\} \to \mathbb{C}^*$$

and extend it to a continuous character

$$\widetilde{\theta}_v : \mathcal{O}_v^* \to \mathbb{C}^*.$$

Now the character

$$\prod_{w\neq\infty}\widetilde{\theta}_w\times\widetilde{\theta}_\infty:\left(\prod_{w\neq\infty}\mathcal{O}_w^*\times L_\infty^*\right)/\mathcal{O}_L^*\to\mathbb{C}^*$$

is continuous. Extend it arbitrarily to a character $\widetilde{\theta}$ of $L^*\backslash\mathbb{A}_L^*$.

We verify the conditions (1)–(3) for $\pi(\widetilde{\theta})$. By construction $\widetilde{\theta}_w$ is unramified for all finite places w not equal to v, and the local extensions $L_w/\mathbb{Q}_l$ are unramified for $q\neq l$, which implies (1). Part (2) follows from Lemma 7.1 of [8]. As we have chosen a non-quadratic character θ'_v in the construction of $\widetilde{\theta}_v$, there exists an element $\gamma\in\mathcal{O}_v^1$ such that $\widetilde{\theta}_v(\gamma)\neq\widetilde{\theta}_v(\gamma^{-1})=\widetilde{\theta}_v(\overline{\gamma})$. The other conditions of Lemma 7.1 of [8] are also satisfied by construction.

Part (3) follows from the multiplicity formulae. The representations π_1 and π_2 are of type (a) and the formula for their multiplicity is given in Proposition 7.3 of [8]. □

Remark 4.2. *The reason for this slightly delicate choice of the local character at the place above q in the above proof is that we are constructing L-packets of an inner form of SL_2, which is not quasi-split. Changing a representation in a global endoscopic packet at a place where the local L-packet is of size two therefore not always changes the multiplicity.*

Now fix an automorphic representation $\pi(\widetilde{\theta})$ as above and such that

$$\widetilde{\theta}_\infty(z)\mapsto(z\overline{z})^{-k_1-1/2}z^{k_1-k_2+1},$$

where $k_1,k_2\in\mathbb{Z}$ and $k_1-k_2+1\geq 2$ is an even integer. In particular,

$$\pi(\widetilde{\theta})_\infty\cong(\mathrm{Sym}^{k_1-k_2}(\mathbb{C}^2)\otimes\mathrm{Nrd}^{k_2})^*.$$

We have two representations π_1 and π_2 as above and we assume that π_2 is automorphic and π_1 is not.

The representation $\pi(\widetilde{\theta})$ shows up in the following eigenvariety: for all $l\neq q$ define $\widetilde{e}_l:=e_{\mathrm{GL}_2(\mathbb{Z}_l)}$ and let $\widetilde{e}_q$ be the special idempotent attached to the Bernstein component defined by the supercuspidal representation $\pi(\widetilde{\theta})_q$. Define $\widetilde{e}=\otimes_l\widetilde{e}_l\in C_c^\infty(\widetilde{G}(\mathbb{A}_f^p),\overline{\mathbb{Q}})$. Let $S=S(\widetilde{e})=\{p,q\}$ and $\widetilde{\mathcal{H}}_S:=\widetilde{\mathcal{H}}_{ur,S}\otimes\widetilde{\mathcal{A}}_p$ as in Section 2. Then by construction $\pi(\widetilde{\theta})$ gives rise to two points on the eigenvariety $\mathcal{D}(\widetilde{e})$, one of which is of critical slope by Lemma 3.3, which we denote again by $\widetilde{x}$.

Let $e_{q,1}$ (respectively $e_{q,2}$) $\in C_c^\infty(G(\mathbb{Q}_q),\overline{\mathbb{Q}})$ be the special idempotent associated with τ_1 (respectively τ_2) and define

$$e_1:=\bigotimes_{l\neq q,p}e_{\mathrm{SL}_2(\mathbb{Z}_l)}\otimes e_{q,1}\in C_c^\infty(G(\mathbb{A}_f^p),\overline{\mathbb{Q}})\text{ and}$$

$$e_2:=\bigotimes_{l\neq q,p}e_{\mathrm{SL}_2(\mathbb{Z}_l)}\otimes e_{q,2}\in C_c^\infty(G(\mathbb{A}_f^p),\overline{\mathbb{Q}}).$$

Theorem 4.3. *There exist points $x_1\in\mathcal{D}(e_1)(\overline{\mathbb{Q}}_p)$ and $x_2\in\mathcal{D}(e_2)(\overline{\mathbb{Q}}_p)$ such that*

$$(\psi_{x_1},\omega(x_1))=(\psi_{x_2},\omega(x_2))=(\psi_{\widetilde{x}}|_{\mathcal{H}_S},\mu(\omega(\widetilde{x}))).$$

Proof. By construction π_2 is an automorphic representation such that $e_2 \cdot (\pi_2)_f^p \neq 0$ and so there is a point $x_2 \in \mathcal{D}(e_2)(\overline{\mathbb{Q}}_p)$ as claimed.

For the existence of x_1 we use the p-adic transfer. Recall the notation of Remark 2.6. We have a Zariski-dense and accumulation set Z' on $\mathcal{D}'(\widetilde{e})$, which is in bijection with the set of pairs

$$\{(\lambda' \circ \psi_{(\widetilde{\pi},\chi)}, (k_1, k_2))\},$$

where $\widetilde{\pi}$ is a p-refined automorphic representation of $\widetilde{G}(\mathbb{A})$ of weight (k_1, k_2) (cf. Section 3.3 of [10]). In particular, $\lambda'(\widetilde{x}) \in Z'$.

Let Π_s be the set of all stable L-packets of $G(\mathbb{A})$ and let

$$Z'_s := \{(\lambda' \circ \psi_{(\widetilde{\pi},\chi)}, (k_1, k_2)) \in Z' \mid \Pi(\widetilde{\pi}) \in \Pi_s\}$$

be the subset of Z' arising from representations $\widetilde{\pi}$ that do not come from a Größen-character. This is well-defined (cf. [10] Section 3.3.1).

Claim: There exists an open affinoid neighbourhood U of $\lambda'(\widetilde{x}) \in \mathcal{D}'(\widetilde{e})(\overline{\mathbb{Q}}_p)$ such that $Z'_s \cap U$ is Zariski-dense and accumulation in U. Indeed choose any open affinoid neighbourhood W of $\lambda'(\widetilde{x})$ and let

$$V := \{x \in W \mid v_p(\psi_x(u_0)) = v_p(\psi_{\lambda'(\widetilde{x})}(u_0)) = 2(k_1 - k_2 + 1)\}.$$

This is an affinoid neighbourhood of $\lambda'(\widetilde{x})$. Choose $U \subset V$ to be an open affinoid neighbourhood of $\lambda'(\widetilde{x})$ with the property that $\omega'(U) \subset \widetilde{\mathcal{W}}$ is open affinoid and the induced morphism $\omega'|_U : U \to \omega'(U)$ is finite and surjective when restricted to any irreducible component of U.

To see that $Z'_s \cap U$ is Zariski-dense and accumulation, let $y := \mu(\widetilde{\omega}(\widetilde{x})) = k_1 - k_2$ and $y' := 2(k_1 - k_2) + 1 \in \mathcal{W}(E)$ and define $Y := (\mu \circ \omega')|_U^{-1}(\{y, y'\})$ to be the fibre, which is a Zariski-closed subspace of U of codimension 1. Let $U' := U \backslash Y$. Then $Z' \cap U' \subset Z'_s$ by Lemma 3.3 and Lemma 3.5 above. But $Z' \cap U'$ is still Zariski-dense and accumulation in U, as we have only removed a Zariski-closed subset of smaller dimension. This proves the claim.

Define $e := \bigotimes_{l \neq q,p} e_{\mathrm{SL}_2(\mathbb{Z}_l)} \otimes e_q \in C_c^\infty(G(\mathbb{A}_f^p), \overline{\mathbb{Q}})$, where e_q is the special idempotent associated with the two Bernstein components defined by the representations τ_1 and τ_2. Then $\widetilde{e}$ and e are Langlands compatible and we have a p-adic transfer as in Theorem 2.5.

By Remark 2.7 we have a closed immersion $\mathcal{D}(e_1) \hookrightarrow \mathcal{D}(e)$, which we base-change along $\mu : \widetilde{\mathcal{W}} \to \mathcal{W}$ to $\iota : \mathcal{D}''(e_1) \hookrightarrow \mathcal{D}''(e)$. Consider the following diagram (cf. Remark 2.6)

$$\begin{array}{ccccccc} & & & & \mathcal{D}''(e_1) & \longrightarrow & \mathcal{D}(e_1)\ . \\ & & & & \big\downarrow \iota & & \big\downarrow \\ \mathcal{D}(\widetilde{e}) & \xrightarrow{\lambda'} & \mathcal{D}'(\widetilde{e}) & \xrightarrow{\xi} & \mathcal{D}''(e) & \longrightarrow & \mathcal{D}(e) \end{array}$$

As ξ and ι are closed immersions of equi-dimensional rigid analytic spaces, their images are a union of irreducible components of $\mathcal{D}''(e)$. We identify $\mathcal{D}'(e)$ and $\mathcal{D}''(e_1)$ with their images in $\mathcal{D}''(e)$, i.e., we consider them as subspaces of $\mathcal{D}''(e)$.

Let T be an irreducible component of $\mathcal{D}'(\widetilde{e})$ containing $\lambda'(\widetilde{x})$. Let U be as above. Then $U \cap Z'_s \cap T$ is still Zariski-dense in $U \cap T$ and so there exists a point $s \in Z'_s \cap T$ and we can also assume that $s \notin T' \cap T$ for any irreducible component $T' \neq T$. As $s \in Z'_s$, s comes from an automorphic representation that gives rise to a stable L-packet for G, which implies that $s \in \mathcal{D}''(e_1)$. Therefore $T \subset \mathcal{D}''(e_1)$ and in particular, $\lambda'(\widetilde{x}) \in \mathcal{D}''(e_1)$. □

Remark 4.4. *One can of course cook up other examples by using more general imaginary quadratic fields L and quaternion algebras B with more ramified primes. For example assume that B is ramified at more places and π' is a representation in an endoscopic L-packet $\Pi(\widetilde{\pi})$, such that π'_l is unramified for all $l \notin S_B$, with $m(\pi') = 0$, and such that $\Pi(\widetilde{\pi}_p)$ has size one. Then one can again use the special idempotents at the bad places to construct idempotents $e \in C_c^\infty(G(\mathbb{A}_f^p), \overline{\mathbb{Q}})$ such that*

$$e \cdot \pi_f^p \neq 0 \text{ for } \pi \in \Pi(\widetilde{\pi}) \text{ if and only if } \pi = \pi'.$$

In fact this trick works as long as π'_l is supercuspidal at all places where it is not unramified.

Corollary 4.5. *In the notation of Theorem 4.3 define $\varphi := \psi_{\widetilde{x}}|_{\mathcal{H}_S}$ and let $n = \mu(\widetilde{\omega}(\widetilde{x})) = k_1 - k_2 \in \mathcal{W}(E)$. The eigenspaces $M(e_1, n)^\varphi$ and $M(e_2, n)^\varphi$ are both non-zero. The Galois representations attached to the eigenforms in these two spaces agree.*

Proof. The Galois representations exist by Lemma 2.2 and depend only on the points on the eigenvariety defined by the eigenforms. But the images of x_1 and x_2 in $\mathcal{D}(e)$, with e as in the proof of Theorem 4.3, agree by construction. □

5. Consequences

Definition 5.1. *Let $\omega : \mathcal{D}(e) \to \mathcal{W}$ (respectively $\widetilde{\omega} : \mathcal{D}(\widetilde{e}) \to \widetilde{\mathcal{W}}$) be an eigenvariety of idempotent type e (respectively $\widetilde{e}$). We call a point $z \in \mathcal{D}(e)(\overline{\mathbb{Q}}_p)$ (resp. $\mathcal{D}(\widetilde{e})(\overline{\mathbb{Q}}_p)$)* classical *if there exists $f \in M(e, \omega(z))^{cl}$ (resp. $M(\widetilde{e}, \widetilde{\omega}(z))^{cl}$) such that $h \cdot f = \psi_z(h) f$ for all $h \in \mathcal{H}_{S(e)}$ (resp. $\widetilde{\mathcal{H}}_{S(\widetilde{e})}$).*

For the group $\widetilde{G}$ we have the following phenomenon.

Proposition 5.2. *Assume $\widetilde{e}'$ and $\widetilde{e}$ in $C_c^\infty(\widetilde{G}(\mathbb{A}_f), \overline{\mathbb{Q}}_p)$ are idempotents with $S(\widetilde{e}') = S(\widetilde{e}) =: S$ and assume for all $l \in S$ the local idempotents $\widetilde{e}'_l$ and $\widetilde{e}_l$ are special idempotents associated to Bernstein components. Assume that $\widetilde{e} * \widetilde{e}' = \widetilde{e}' = \widetilde{e}' * \widetilde{e}$ so that we have a closed immersion $h : \mathcal{D}(\widetilde{e}') \hookrightarrow \mathcal{D}(\widetilde{e})$. Assume $z \in \mathcal{D}(\widetilde{e}')(\overline{\mathbb{Q}}_p)$ is such that $h(z)$ is classical. Then z is classical.*

Proof. We have to show that there exists a classical automorphic eigenform $f \in M(\widetilde{e}', \widetilde{\omega}(z))^{cl}$ with system of Hecke eigenvalues ψ_z. By construction of the eigenvarieties there exists an overconvergent eigenform $f_{oc} \in M(\widetilde{e}', \widetilde{\omega}(z))$ with system of Hecke eigenvalues given by ψ_z. By assumption the point $h(z)$ is classical, so there exists an eigenform $g \in M(\widetilde{e}, \widetilde{\omega}(z))^{cl}$ for the same system of Hecke eigenvalues $\psi_{h(z)} = \psi_z$. In particular, we have a p-refined automorphic representation $\widetilde{\pi}_{h(z)}$

giving rise to $h(z)$. Both eigenvarieties $\mathcal{D}(\widetilde{e}')$ and $\mathcal{D}(\widetilde{e})$ carry pseudo-representations T' and T (see Prop. 3.10 of [10]) and $T' = h^* \circ T$, where $h^* : \mathcal{O}(\mathcal{D}(\widetilde{e})) \to \mathcal{O}(\mathcal{D}(\widetilde{e}'))$ denotes the homomorphism induced by h. For $l \in S$ let I_l be the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q})$. By Lemma 7.8.18 of [1], $T|_{I_l}$ is constant on the connected components of $\mathcal{D}(\widetilde{e})$. Let $x \in \mathcal{D}(\widetilde{e}')$ be a classical point on the same connected component as z. Then $T'_x|_{I_l} = T'_{h(z)}|_{I_l} = T_z|_{I_l}$. Local-global compatibility, compatibility of the local Jacquet–Langlands transfer with twists and the inertial local Langlands correspondence (see Appendix 1.2 of [2]) imply that the local components $(\widetilde{\pi}_{h(z)})_l$ and $(\widetilde{\pi}_x)_l$ are in the same Bernstein component. Therefore $\widetilde{e}'_l \cdot (\widetilde{\pi}_{h(z)})_l \neq 0$. □

The situation for eigenvarieties of the group G is different. First of all we have the following result:

Proposition 5.3. *In the notation of Theorem 4.3, the point $x_1 \in \mathcal{D}(e_1)(\overline{\mathbb{Q}}_p)$ is not classical.*

Proof. Assume x_1 is classical. Then there exists an automorphic representation π of $G(\mathbb{A})$ such that

$$e_1 \cdot \pi_f^p \neq 0, \tag{2}$$

in particular $\pi_l^{\mathrm{SL}_2(\mathbb{Z}_l)} \neq 0$ for all $l \notin \{p, q\}$. The system of Hecke eigenvalues $\psi_{\widetilde{x}}|_{\mathcal{H}_{ur,S(\widetilde{e})}}$ determines the representation π_l with $\pi_l^{\mathrm{SL}_2(\mathbb{Z}_l)} \neq 0$ and the local L-packet $\Pi_l = \Pi(\pi(\widetilde{\theta})_l)$ uniquely. But this implies $\pi \in \Pi(\pi(\widetilde{\theta}))$ (cf. Theorem 4.1.2 of [11]). Condition (2) implies that $\pi_q = \tau_1$, so the only choice left might be at p. But by construction $\Pi(\pi(\widetilde{\theta})_p) = \{\pi_{1,p}\}$ is a singleton and therefore $\pi \cong \pi_1$. But $m(\pi_1) = 0$. □

Remark 5.4. *It is obvious that one cannot produce non-classical points starting from Größencharacters of an imaginary quadratic field L in which p is inert. There are multiple reasons for this. For example, note that any 'candidate for x_1' that one would end up constructing would automatically be classical by the classicality theorem. The critical slope for u_0 is given by $2(k_1 - k_2 + 1)$ and Lemma 3.5 implies that the slope of the u_0-eigenvalue of any candidate is $k_1 - k_2 + 1$.*

Corollary 5.5. *Let $\mathcal{D}(e)$ be an eigenvariety of idempotent type e for G and assume $z \in \mathcal{D}(e)(\overline{\mathbb{Q}}_p)$ is a point whose system of Hecke eigenvalues ψ_z comes from a classical automorphic representation $\widetilde{\pi}$ of $\widetilde{G}(\mathbb{A})$. Then z is not necessarily classical.*

Remark 5.6. *Note however that one can always enforce classicality by passing to a suitable idempotent (e.g., the idempotent attached to a sufficiently small compact open subgroup $K \subset G(\mathbb{A}_f^p)$). In our example, the image of the non-classical point x_1 under the map $\mathcal{D}(e_1) \hookrightarrow \mathcal{D}(e)$ from the proof of Theorem 4.3 is classical. The analogue of Proposition 5.2 for G is therefore false.*

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