ON ENDOSCOPIC p-ADIC AUTOMORPHIC FORMS FOR SL_2

JUDITH LUDWIG

Abstract. We show the existence of some non-classical cohomological p-adic automorphic eigenforms for SL_2 using endoscopy and the geometry of eigenvarieties. These forms seem to account for some non-automorphic members of classical global L-packets.

1. Introduction

Let $\Pi(\theta)$ be an endoscopic L-packet of $\mathrm{SL}_2/\mathbb{Q}$ associated to an algebraic character

$$\theta : \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m(\mathbb{A}) \to \mathbb{C}^*,$$

where $F/\mathbb{Q}$ is an imaginary quadratic extension in which p splits. For a sufficiently small tame level $K^p \subset \mathrm{SL}_2(\mathbb{A}_f^p)$ an automorphic representation $\pi \in \Pi(\theta)$ gives rise to a point x of critical slope on an eigenvariety of tame level K^p.

Let $E(K^p, \mathfrak{m})$ be such an the eigenvariety, built from the suitably localized completed cohomology groups $\widetilde{H}^1(K^p)_{\mathfrak{m}}$ as in [8]. Emerton's eigenvariety comes equipped with a coherent module $\mathcal{M}$ of p-adic automorphic forms of tame level K^p. If z is a point on $E(K^p, \mathfrak{m})$ coming from a classical automorphic representation, the fibre $\mathcal{M}_{\overline{z}} := \mathcal{M}_z \otimes k(z)$ contains a subspace

$$\mathcal{M}_{\overline{z}}^{cl} \subset \mathcal{M}_{\overline{z}}$$

of classical automorphic forms. Our main theorem concerns the difference between this classical subspace and the whole fibre at critical points x coming from endoscopic L-packets as above. Under some technical assumptions on the L-packet (see *Hypothesis* $(\star)$ in Section 6) we can prove that these spaces differ. The main result of this paper is the following.

Theorem (Theorem 6.5). *Let $\Pi(\theta)$ be an endoscopic L-packet satisfying Hypothesis $(\star)$. Then there exists a tame level K^p such that the fibre $\mathcal{M}_{\overline{x}}$ at the critical point $x \in E(K^p, \mathfrak{m})(\overline{\mathbb{Q}}_p)$ contains non-classical p-adic automorphic forms, i.e.*

$$\mathcal{M}_{\overline{x}}/\mathcal{M}_{\overline{x}}^{cl} \neq 0.$$

Our motivation for this theorem is twofold. Firstly, we would like to understand how endoscopy works in the setting of the p-adic Langlands programme. As a first step we would like to know whether *non automorphic* members of classical global L-or A-packets play a role in the p-adic setting. We ask the following vague question: Let $G/\mathbb{Q}$ be a symplectic, orthogonal or unitary group. Let Π be a global

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endoscopic cohomological L- or A-packet and let $\pi \in \Pi$ be an element such that the multiplicity $m(\pi)$ in the automorphic spectrum of G is equal to zero. Then does π occur p-adically? Is there a point on a suitable eigenvariety and a non-classical p-adic eigenform f for the system of Hecke eigenvalues determined by π such that f accounts for π? The non-classical forms appearing in our theorem seem to provide a p-adic account for non-automorphic representations in $\Pi(\theta)$, although we caution the reader that this is to be understood only in a vague sense.

The second motivation to study the spaces $\mathcal{M}_{\overline{x}}$ of the theorem comes from a related result for the group GL_2. There is a strong relationship between automorphic representations of the groups GL_2 and SL_2; every member of a global L-packet of SL_2 occurs in the restriction to SL_2 of an automorphic representation of GL_2. In the language of modular forms, endoscopic cohomological L-packets are those arising from modular forms with complex multiplication by an imaginary quadratic field. Now consider points of critical slope on a Coleman-Mazur eigencurve that come from such modular forms. These points are interesting from a geometric point of view as the map down to weight space ramifies there. Equivalently but more importantly for us, the corresponding generalized eigenspace of overconvergent forms contains non-classical forms (see Remark 6.6). Our theorem can be seen as an analogue for SL_2 of this fact, with the notable difference that our theorem concerns honest eigenspaces.

Although the situations for GL_2 and SL_2 are clearly related, we cannot deduce our result from the GL_2-situation. Instead we use endoscopy and geometry on the SL_2-eigencurve to prove our result. We briefly explain the strategy and the key ideas of the proof.

Strategy: Show that a point x as in the theorem has an open affinoid neighbourhood $U \subset E(K^p, \mathfrak{m})$ that contains a Zariski-dense set of classical *stable* points, i.e., points arising from L-packets Π that are stable. Then compare the dimension of $\mathcal{M}^{cl}_{\overline{x}}$ to the dimension of the classical subspace $\mathcal{M}^{cl}_{\overline{z}}$ of the fibre at a stable point z and show that the rank at stable points is larger. This implies the theorem as the fibre rank of a coherent sheaf is semi-continuous.

We briefly explain why there is more contribution from automorphic representations at stable points than at endoscopic points. For that first note that the classical subspace $\mathcal{M}^{cl}_{\overline{z}}$ of the fibre $\mathcal{M}_{\overline{z}}$ at any classical point z of weight k decomposes into a direct sum

$$\mathcal{M}^{cl}_{\overline{z}} \cong \bigoplus_{\pi \in X(z)} m(\pi)\ M_\pi \otimes H^1_{rel.Lie}(\mathfrak{g}, \mathrm{SO}_2(\mathbb{R}), \pi_\infty \otimes \mathrm{Sym}^k(\mathbb{C}^2))$$

indexed over a certain subset $X(z)$ of the global L-packet $\Pi(z)$ defined by the point z. The number $m(\pi)$ is the multiplicity of π in the cuspidal automorphic spectrum and M_π is the tensor product of a subspace of the smooth Jacquet module $J_B(\pi_p)$ and the spaces $\pi_l^{K_l}$, with l running through the set of primes where $K_l \neq \mathrm{SL}_2(\mathbb{Z}_l)$.

Crucial observation: Ignoring the spaces M_π, the classical subspace of the fibre at x has half as many non-zero summands as the space at a stable point z. The reason for this is that in a stable packet all elements are automorphic, contrary to the situation for the L-packet $\Pi(\theta)$. The archimedean L-packet $\Pi(\theta)_\infty = \{D^+_{k(\theta)}, D^-_{k(\theta)}\}$ is a discrete series L-packet of size two and given $\tau_f \in \Pi(\theta)_f$ exactly one of the representations

$$\tau_f \otimes D^+_{k(\theta)}, \tau_f \otimes D^-_{k(\theta)}$$

is automorphic. So if we manage to control the contributions of the terms M_π in a family, in other words if we can find some tame level that minimizes the terms M_π for $\pi \in X(x) \subset \Pi(\theta)$, then we get the estimate that we seek. This is achieved using on the one hand the theory of newforms and conductors for SL_2 as developed in [15]. The main technical result here is Proposition 6.4. The second ingredient is a rigidity result on the behaviour of L-packets in families (see Proposition 5.10).

In [17], we constructed non-classical points on certain eigenvarieties for a definite inner form G of SL_2 using endoscopy and a p-adic version of the Labesse-Langlands transfer [16]. We used a local L-packet of size two to change multiplicities in a global L-packet, however in [17] that auxiliary prime was non-archimedean. In this paper we explore the fact that for the split group $\mathrm{SL}_2(\mathbb{R})$ the discrete series L-packets are of size two. The advantage of the method of this paper is that it should work for other groups as well.

Outline of the paper. After recalling some background on the cohomology of the symmetric space of SL_2 in Section 2, we summarize results of [15] on conductors of smooth representations of $\mathrm{SL}_2(\mathbb{Q}_l)$. The important results are the dimension formulas of Proposition 3.4. Section 4 is on the local symmetric square lifting, we need Proposition 4.2 to prove the rigidity result 5.10 in the next section. The rest of Section 5 sets up the eigenvarieties we use. We then prove our main theorem in the last section.

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2. Background

Let $\mathfrak{g} = \mathfrak{sl}_2$ be the Lie-algebra of $\mathrm{SL}_2(\mathbb{R})$ and define $K_\infty = \mathrm{SO}_2(\mathbb{R})$. Let $W_\mathbb{C} := \mathrm{Sym}^k(\mathbb{C}^2) \cong \check{W}_\mathbb{C}$ be the algebraic representation of SL_2 of highest weight k.

Below we need the following result on the relative Lie-algebra cohomology.

Lemma 2.1 ([14, Section 2.1])**.**

$$H^1_{rel.Lie}(\mathfrak{g}, K_\infty, W_\mathbb{C} \otimes \pi_\infty) = \begin{cases} \mathbb{C}, & \textit{if } \pi_\infty = D^\pm_{k+1} \\ 0, & \textit{otherwise.} \end{cases}$$

Here D^+_{k+1} and D^-_{k+1} denote the holomorphic and antiholomorphic discrete series representation of weight $k+1$, i.e., the infinite-dimensional constituents of the non-unitary principal series representation I_{k+1} of $\mathrm{SL}_2(\mathbb{R})$ associated to the character $t \mapsto t^{k+1}(\mathrm{sgn}(t))^k$ of $\mathbb{R}^*$, c.f. [14] Section 2.1.

For a compact open subgroup $K = \prod_l K_l \subset \mathrm{SL}_2(\mathbb{A}_f)$ consider the symmetric space

$$Y(K) = \mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A}) / K\, \mathrm{SO}_2(\mathbb{R}).$$

Let $\mathcal{W}_k$ denote the local system on $Y(K)$ attached to $W_{\mathbb{C}}$ and consider the singular cohomology group

$$H^1(K, W_{\mathbb{C}}) := H^1(Y(K), \mathcal{W}_k).$$

Recall that the parabolic cohomology[1]

$$H^1_{\mathrm{par}}(K, W_{\mathbb{C}}) := H^1_{\mathrm{par}}(Y(K), \mathcal{W}_k)$$

is defined as the image of the natural map

$$H^1_c(Y(K), \mathcal{W}_k) \to H^1(Y(K), \mathcal{W}_k).$$

By [20, Corollary 2.3] parabolic cohomology and cuspidal cohomology agree for the groups $\mathrm{SL}_2 /\mathbb{Q}$, we have a decomposition

$$H^1_{\mathrm{par}}(K, W_{\mathbb{C}}) = \bigoplus_{\substack{\pi \ adm.\ rep. \\ of\, \mathrm{SL}_2(\mathbb{A})}} m(\pi)\ \pi_f^K \otimes H^1_{rel.Lie}(\mathfrak{g}, K_\infty, W_{\mathbb{C}} \otimes \pi_\infty), \tag{1}$$

where $m(\pi)$ is the multiplicity of π in the cuspidal automorphic spectrum.

Let $S(K)$ be a finite set of primes such that $K_l = \mathrm{SL}_2(\mathbb{Z}_l)$ for all $l \notin S(K)$. The unramified Hecke algebra

$$\mathcal{H}^{\mathrm{ur}} = \bigotimes{}'_{l \notin S(K)} C_c^\infty(\mathrm{SL}_2(\mathbb{Q}_l) \,//\, \mathrm{SL}_2(\mathbb{Z}_l))$$

acts on $H^1(K, W_{\mathbb{C}})$ and $H^1_{\mathrm{par}}(K, W_{\mathbb{C}})$. We recall that for a *cuspidal* automorphic representation π of $\mathrm{SL}_2(\mathbb{A})$ occurring in $\varinjlim_{K'} H^1(K', W_{\mathbb{C}})$ with $\pi_f^K \neq 0$ and unramified system of Hecke eigenvalues $\lambda : \mathcal{H}^{\mathrm{ur}} \to \mathbb{C}^*$, we have an isomorphism

$$H^1(K, W_{\mathbb{C}})^\lambda \cong H^1_{\mathrm{par}}(K, W_{\mathbb{C}})^\lambda.$$

We also remind the reader that any cuspidal automorphic representation π of SL_2 occurs with multiplicity one in the space of automorphic forms ([19, Theorem 4.1.1]).

For an irreducible smooth representation π_p of $\mathrm{SL}_2(\mathbb{Q}_p)$ let $J_B(\pi_p)$ be its Jacquet-module with respect to the upper triangular Borel B. It is a smooth representation of the torus $T \subset \mathrm{SL}_2(\mathbb{Q}_p)$ of diagonal matrices. For a character $\chi : T \to \mathbb{C}^*$ we consider its eigenspace $J_B(\pi_p)^\chi$.

Lemma 2.2. *Let $\pi_p \cong \mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{Q}_p)}(\chi)$ be an irreducible unramified principal series representation. Then $J_B(\pi_p)^\chi$ is one-dimensional.*

Proof. The principal series representation is irreducible, therefore χ is not a quadratic character, i.e. $\chi \neq \chi^{-1}$. The Jacquet-module $J_B(\mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{Q}_p)}(\chi))$ is two-dimensional and its semi-simplification as a $\mathbb{C}[T]$-module is given by

$$J_B(\mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{Q}_p)}(\chi))^{ss} \cong \delta_B^{1/2}\chi \oplus \delta_B^{1/2}\chi^{-1},$$

where δ_B is the modulus character. □

[1] This is often called interior cohomology.

3. Newforms for $\mathrm{SL}_2(\mathbb{Q}_l)$

In Section 6 we need some of the results of [15] which we summarize here. Let $l \neq 2$ be a prime number. Given an irreducible smooth representation $\widetilde{\pi}$ of $\mathrm{GL}_2(\mathbb{Q}_l)$ its restriction to $\mathrm{SL}_2(\mathbb{Q}_l)$ breaks up into a finite direct sum of irreducible smooth representations of $\mathrm{SL}_2(\mathbb{Q}_l)$, each occurring with multiplicity one. A local L-packet of $\mathrm{SL}_2(\mathbb{Q}_l)$ is a finite set of irreducible smooth representations of $\mathrm{SL}_2(\mathbb{Q}_l)$ arising in this way. They are of size $1, 2$ or 4. We refer to [13] for details.

Definition 3.1. *(1) We say a local L-packet of $\mathrm{SL}_2(\mathbb{Q}_l)$ is unramified, if it contains a representation π_l with $\pi_l^{\mathrm{SL}_2(\mathbb{Z}_l)} \neq 0$. Otherwise we call it ramified.*
(2) A local L-packet Π of $\mathrm{SL}_2(\mathbb{Q}_l)$ is called supercuspidal if one (equiv. all) $\pi \in \Pi$ are supercuspidal.

Remark 3.2. *In an unramified local L-packet the element π_l with $\pi_l^{\mathrm{SL}_2(\mathbb{Z}_l)} \neq 0$ is unique and we denote it by π_l^0. Supercuspidal local L-packets are of cardinality two or four. As we have assumed $l \neq 2$, all supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_l)$ can be constructed from characters $\chi : E^* \to \mathbb{C}^*$ of quadratic extensions $E/\mathbb{Q}_l$ which do not factor through the norm $\mathrm{N}^E_{\mathbb{Q}_l}$. We write $\pi(\chi)$ for this representation and refer to Chapter 39 of [3] for the construction. Let $\overline{\chi}$ denote the conjugate of χ under the non-trivial element of $\tau \in \mathrm{Gal}(E/\mathbb{Q}_l)$, so $\overline{\chi}(x) = \chi(\tau x)$. The L-packet $\Pi(\pi(\chi))$ defined by $\pi(\chi)$ is of size two unless $\chi\overline{\chi}^{-1}$ is a quadratic character, when it is of size four.*

For $m \geq 0$ consider the compact open subgroups of $\mathrm{SL}_2(\mathbb{Z}_l)$

$$K_1(m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_l) | c \equiv 0, d \equiv 1 \mod l^m \right\}$$

$$\text{and } K_0(m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_l) | c \equiv 0 \mod l^m \right\}.$$

The group $K_1(m)$ is normal in $K_0(m)$. The quotient $K_0(m)/K_1(m)$ is isomorphic to $\mathbb{Z}_l^*/(1 + l^m\mathbb{Z}_l)$. Define $\alpha := \begin{pmatrix} l & \\ & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_l)$ and let

$$K_0'(m) := \alpha^{-1} K_0(m) \alpha \subset \mathrm{SL}_2(\mathbb{Q}_l)$$

be the conjugate subgroup.

Let $\eta : \mathbb{Z}_l^* \to \mathbb{C}^*$ be a character and let $c(\eta)$ be its conductor. Then for any $m \geq c(\eta)$, η defines a character of $K_0(m)$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \eta(d)$, which we still denote by η and similarly for $K_0'(m)$. Let π be an irreducible smooth representation of $\mathrm{SL}_2(\mathbb{Q}_l)$. Define

$$\pi_\eta^{K_0(m)} := \left\{ v \in \pi : \pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) v = \eta(d) v \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(m) \right\}$$

and $\pi_\eta^{K_0'(m)}$ analogously.

Let ω_π denote the central character of π. Note that if $\eta(-1) \neq \omega_\pi(-1)$, then $\pi_\eta^{K_0(m)} = 0$ and $\pi_\eta^{K_0'(m)} = 0$. We remark here that the central characters of any two elements in a local L-packet agree. This is obvious from the definition of a local L-packet as the set of representations occurring in the restriction of an irreducible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_l)$.

The authors of [15] define *the conductor* $c(\pi)$ of an irreducible smooth representation π of $\mathrm{SL}_2(\mathbb{Q}_l)$ as

$$c(\pi) := \min_{\eta:\eta(-1)=\omega_\pi(-1)} \min\{m \geq 0 : \pi_\eta^{K_0(m)} \neq 0 \text{ or } \pi_\eta^{K_0'(m)} \neq 0\},$$

where η runs over all characters of $\mathbb{Z}_l^*$ with $\eta(-1) = \omega_\pi(-1)$.

Proposition 3.3 ([15, Theorem 3.4.1])**.** *Let π be an irreducible admissible representation of $\mathrm{SL}_2(\mathbb{Q}_l)$. The conductor $c(\pi)$ depends only on the L-packet containing π. It agrees with the conductor $c(\widetilde{\pi})$ of a minimal representation $\widetilde{\pi}$ of $\mathrm{GL}_2(\mathbb{Q}_l)$ such that π occurs in the restriction $\widetilde{\pi}|_{\mathrm{SL}_2(\mathbb{Q}_l)}$.*

Here a representation $\widetilde{\pi}$ of $\mathrm{GL}_2(\mathbb{Q}_l)$ is called minimal if $c(\widetilde{\pi} \otimes \chi) \geq c(\widetilde{\pi})$ for all characters χ of $\mathbb{Q}_l^*$. The proposition lets us talk about the conductor of a local L-packet Π and write $c(\Pi)$ for it.

In [15], the authors study the various spaces $\pi_\eta^{K_0(m)}$ and $\pi_\eta^{K_0'(m)}$ for π varying in a given L-packet Π and all $m \geq c(\Pi)$. They determine the dimensions of these spaces depending on the type of the L-packet. Contrary to the $\mathrm{GL}_2(\mathbb{Q}_l)$ case (see [4]), the first non-zero space $\pi_\eta^{K_0(c(\pi))}$ or $\pi_\eta^{K_0'(c(\pi))}$ is not necessarily always one-dimensional.

We summarize the results of [15] for supercuspidal L-packets. For that let Π be a L-packet arising from an irreducible supercuspidal representation $\widetilde{\pi}$ of $\mathrm{GL}_2(\mathbb{Q}_l)$. One has to distinguish between two kinds of such L-packets. The first kind (called *unramified supercuspidal L-packet* in [15]) is given by representations $\widetilde{\pi}$ that are compactly induced from a representation $\widetilde{\sigma}$ of $\widetilde{Z}\,\mathrm{GL}_2(\mathbb{Z}_l)$. Here $\widetilde{Z}$ denotes the center of $\mathrm{GL}_2(\mathbb{Q}_l)$. The second kind (called *ramified supercuspidal L-packet* in [15]) is given by representations $\widetilde{\pi}$ that are compactly induced from the normalizer $N_{\mathrm{GL}_2(\mathbb{Q}_l)}\widetilde{I}$ of the standard Iwahori $\widetilde{I}$ of $\mathrm{GL}_2(\mathbb{Q}_l)$ of upper triangular matrices mod l. Supercuspidal L-packets of the second kind are always of size two, those of the first kind can be of size four, namely when the representation $\widetilde{\sigma}$ is of level one in the sense of Definition 3.3.1 in [15]. But when this level is greater than or equal to two, the resulting L-packet is of size two. We will avoid local packets of size four below.

Proposition 3.4. *Let $\Pi = \{\pi_1, \pi_2\}$ be a supercuspidal L-packet of cardinality two with conductor $c(\Pi)$. Let n be $\lfloor c(\Pi)/2 \rfloor$. Let η be a character of $\mathbb{Z}_l^*$ with $c(\eta) \leq n$. Then*

(1) For $\pi \in \Pi$, $\dim(\pi_\eta^{K_0(c(\Pi))}) \in \{0,1,2\}$. If Π is of the first kind, we have

$$\dim((\pi_1)_\eta^{K_0(c(\Pi))}) = 0 \text{ and } \dim((\pi_2)_\eta^{K_0(c(\Pi))}) = 2$$

or vice versa. If Π is of the second kind, then

$$\dim((\pi_1)_\eta^{K_0(c(\Pi))}) = \dim((\pi_2)_\eta^{K_0(c(\Pi))}) = 1.$$

In particular we always have

$$\sum_{\pi\in\Pi}\dim(\pi_\eta^{K_0(c(\Pi))}) = 2.$$

(2) More generally for any $m \geq c(\Pi)$,

$$\sum_{\pi\in\Pi}\dim(\pi_\eta^{K_0(m)}) = 2(m - c(\Pi) + 1).$$

Proof. This follows from [15, Proposition 3.3.4] and [15, Proposition 3.3.8]. □

Note also that for a smooth irreducible representation π of $\mathrm{SL}_2(\mathbb{Q}_l)$

$$\pi^{K_1(m)} = \bigoplus_\eta \pi_\eta^{K_0(m)},$$

where η runs through all characters of $K_0(m)/K_1(m)$. This implies that if $\pi^{K_1(m)} \neq 0$ for some $m \geq 0$, then the conductor $c(\pi)$ of π is less than or equal to m.

4. Symmetric square lifting

We recall some background on the local symmetric square lifting as constructed by Gelbart and Jacquet in [10]. In this section, F is a local non-archimedean field of residue characteristic not 2, with residue field $\mathbb{F}$ of size q.

Lemma 4.1 ([10, Prop. 3.3])**.** *Let σ be an irreducible admissible representation of $\mathrm{GL}_2(F)$. Then there exists an irreducible admissible representation π of $\mathrm{GL}_3(F)$ unique up to isomorphism, called the lift of σ, such that*

(1) the central character of π is trivial,
(2) π is self-dual,
(3) for any character χ of F^*,

$$L(s, \pi\otimes\chi) = L(s, (\sigma\otimes\chi)\times\widetilde{\sigma})/L(s,\chi) \text{ and}$$

$$\varepsilon(s, \pi\otimes\chi;\psi) = \varepsilon(s, (\sigma\otimes\chi)\times\widetilde{\sigma};\psi)/\varepsilon(s,\chi;\psi).$$

For the definition of L-and ε-factors see [11]. We give the recipe for the lift of σ. In the following, $B \subset \mathrm{GL}_3(F)$ denotes the Borel subgroup of upper triangular matrices, and $P \subset \mathrm{GL}_3(F)$ the parabolic subgroup of block form $(2, 1)$.

Case (I): $\sigma = \mathrm{Ind}(\mu_1, \mu_2)$ for two characters $\mu_i : F^* \to \mathbb{C}^*$, $i = 1, 2$. Write $\mu_i = \chi_i|.|_F^{t_i}$ where $|x|_F = q^{-v(x)}$, $t_i \in \mathbb{R}$, and $\chi_i : F^* \to \mathbb{C}^*$ satisfies $\chi\overline{\chi} = 1$. If $t_1 - t_2 = 0$, the representation

$$\pi := \mathrm{Ind}_B^{\mathrm{GL}_3(F)}(\mu_1\mu_2^{-1}, 1, \mu_2\mu_1^{-1})$$

is irreducible and a lift of σ. If $t_1 - t_2 \neq 0$, we may assume $t_1 > t_2$. Then the representation $\mathrm{Ind}_B^{\mathrm{GL}_3(F)}(\mu_1\mu_2^{-1}, 1, \mu_2\mu_1^{-1})$ admits a maximal subrepresentation τ and the lift of σ is given by the quotient

$$\pi := \mathrm{Ind}_B^{\mathrm{GL}_3(F)}(\mu_1\mu_2^{-1}, 1, \mu_2\mu_1^{-1})/\tau.$$

Case(II): Assume σ is a supercuspidal representation, so $\sigma = \pi(\chi)$ for some character $\chi : E^* \to \mathbb{C}^*$ of a quadratic extension E/F. As before let $\overline{\chi}$ denote the conjugate of χ under the non-trivial element of $\tau \in \mathrm{Gal}(E/F)$, so $\overline{\chi}(x) = \chi(\tau x)$. Define $\mu := \chi\overline{\chi}^{-1}$ and let $\pi(\mu)$ be the representation of $\mathrm{GL}_2(F)$ attached to μ via the Weil-representation. Then the lift of σ is given by

$$\pi = \mathrm{Ind}_P^{\mathrm{GL}_3(F)}(\pi(\mu), \varpi_{E/F}),$$

where $\varpi_{E/F} : F^* \to \mathbb{C}^*$ is the character associated to E/F by local class field theory.

We recall that the representation $\pi(\mu)$ is supercuspidal if and only if μ does not factor through the norm N_F^E, i.e., if and only if $\chi\overline{\chi}^{-1}$ is not a quadratic character.

Case(III): Finally, let σ be a special representation, i.e.,

$$\sigma \cong \mathrm{St} \otimes \chi$$

is a twist of the Steinberg representation of $\mathrm{GL}_2(F)$ by a character χ of F^*. Then the lift of σ is given by the Steinberg representation π of $\mathrm{GL}_3(F)$. It is the square integrable component of

$$\mathrm{Ind}_B^{\mathrm{GL}_3(F)}(|\cdot|, 1, |\cdot|^{-1}).$$

By construction the lifts of σ and any twist $\sigma \otimes \chi$ agree. Therefore we get an induced map

$$\begin{aligned} \{L\text{-packets of } \mathrm{SL}_2(F)\} &\to \{\text{irreducible smooth representations of } \mathrm{GL}_3(F)\} \\ \Pi(\sigma) &\mapsto \pi, \end{aligned}$$

that sends the L-packet $\Pi(\sigma)$ defined by a representation σ of $\mathrm{GL}_2(F)$ to the lift π of σ.

Recall from Remark 3.2 that the L-packet $\Pi(\sigma)$ defined by a supercuspidal representation $\sigma \cong \pi(\chi)$ is of size two unless $\chi\overline{\chi}^{-1}$ is a quadratic character, in which case it is of size four.

Proposition 4.2. *Let σ and σ' be irreducible admissible representations of $\mathrm{GL}_2(F)$ and assume $\sigma = \pi(\chi)$ is supercuspidal and comes from a character χ such that $\chi\overline{\chi}^{-1}$ is not quadratic. Assume the lifts π of σ and π' of σ' are in the same Bernstein component. Then σ' is supercuspidal and defines an L-packet of $\mathrm{SL}_2(F)$ which is of size two.*

Proof. By the previous discussion we see that π is given by $\mathrm{Ind}_P^{\mathrm{GL}_3(F)}(\pi(\mu), \varpi_{E/F})$ for a supercuspidal representation $\pi(\mu)$. As the parabolic P of block form $(2, 1)$ and the Borel subgroup B are not conjugate, the representation π' has to be of the form

$$\pi' \cong \mathrm{Ind}_P^{\mathrm{GL}_3(F)}(\pi(\mu'), \varpi_{E'/F})$$

for a supercuspidal representation $\pi(\mu')$ which satisfies $\pi(\mu) \cong \pi(\mu') \otimes \eta$ for some unramified character η of F^*. By the above description of the lifts we see that this is only possible if $\sigma' = \pi(\chi')$ is also supercuspidal and χ' is such that $\chi'\overline{\chi}'^{-1}$ is not quadratic. $\square$

5. Eigenvarieties a la Emerton

Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}_q$. Fix an embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$ as well as an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. The main reference for this section is [8].

Fix a tame level, i.e., a compact open subgroup

$$K^p = \prod_l K_l \subset \mathrm{SL}_2(\mathbb{A}_f^p)$$

and let $S(K^p)$ be the minimal set of primes l, such that $K_l = \mathrm{SL}_2(\mathbb{Z}_l)$ for all $l \notin S(K^p)$, $l \neq p$.

We write $\mathcal{H}(K^p) := C_c^\infty(\mathrm{SL}_2(\mathbb{A}_f^p) \mathbin{/\!/} K^p)$ for the prime to p Hecke algebra over E of level K^p, where the symbol $/\!/$ means left-and right invariance. It decomposes

$$\mathcal{H}(K^p) \cong \mathcal{H}(K^p)^{\mathrm{ram}} \otimes_E \mathcal{H}(K^p)^{\mathrm{ur}},$$

where

$$\mathcal{H}(K^p)^{\mathrm{ur}} = \bigotimes{}'_{l \notin S(K^p) \cup \{p\}} C_c^\infty(\mathrm{SL}_2(\mathbb{Q}_l) \mathbin{/\!/} \mathrm{SL}_2(\mathbb{Z}_l))$$

and $\mathcal{H}(K^p)^{\mathrm{ram}} = C_c^\infty(\mathrm{SL}_2(\mathbb{A}_{S(K^p)}) \mathbin{/\!/} K_S)$.

For R equal to either $\mathcal{O}, \mathcal{O}/\varpi^s\mathcal{O}$ for some $s > 0$ or to E and $i \geq 0$, define

$$H^i(K^p, R) := \varinjlim_{K_p} H^i(Y(K_pK^p), R),$$

where the direct limit runs over all compact open subgroups $K_p \subset \mathrm{SL}_2(\mathbb{Q}_p)$ and as above

$$Y(K_pK^p) = \mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A})/K_pK^p\mathrm{SO}_2(\mathbb{R})$$

is the symmetric space of level K_pK^p. Recall that the completed cohomology of tame level K^p is defined as

$$\widetilde{H}^i(K^p, \mathcal{O}) := \varprojlim_s H^i(K^p, \mathcal{O}/\varpi^s\mathcal{O}) = \varprojlim_s \varinjlim_{K_p} H^i(Y(K_pK^p), \mathcal{O}/\varpi^s\mathcal{O}).$$

Then

$$\widetilde{H}^i(K^p) := \widetilde{H}^i(K^p, \mathcal{O}) \otimes_{\mathcal{O}} E$$

is an E-Banach space equipped with an action of $\mathcal{H}(K^p)$ as well as an admissible continuous action of the locally analytic group $\mathrm{SL}_2(\mathbb{Q}_p)$ in the sense of [6]. Taking the limit

$$\widetilde{H}^i := \varinjlim \widetilde{H}^i(K^p),$$

we get a locally convex topological E-vector space together with an admissible continuous action of $\mathrm{SL}_2(\mathbb{A}_f)$. We recover the previous space by taking K^p-fixed vectors:

$$(\widetilde{H}^i)^{K^p} = \widetilde{H}^i(K^p).$$

The only interesting cohomology index for the group $\mathrm{SL}_2/\mathbb{Q}$ is $i = 1$. As for K_p small enough the $Y(K_pK^p)$ are connected open Riemann surfaces, the cohomology groups $H^i(K^p, R)$ vanish for all $i \geq 2$. Furthermore $H^0(K^p, R) \cong R$. For any inclusion $K_p \subset K_p'$ of compact open subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$, the transition maps

$$H^0(Y(K_p'K^p), R) \cong R \to H^0(Y(K_pK^p), R) \cong R$$

are the identity, therefore

$$\widetilde{H}^0(K^p,\mathcal{O}) \cong \mathcal{O}.$$

Furthermore both $\widetilde{H}^0(K^p,\mathcal{O})$ and $\widetilde{H}^1(K^p,\mathcal{O})$ agree with the p-adic completions

$$\widehat{H}^i(K^p,\mathcal{O}) := \varprojlim_s H^i(K^p,\mathcal{O})/\varpi^s,$$

and are $\mathcal{O}$-torsion free.

The abstract "unramified" Hecke algebra over $\mathcal{O}$, i.e., the polynomial algebra

$$\mathbb{H} := \mathcal{O}[t_l : l \notin S(K^p) \cup \{p\}]$$

acts on the spaces $H^1(Y(K_pK^p),\mathcal{O})$.

Let $\widetilde{\mathbb{H}}(K_p)$ be the image of $\mathbb{H}$ in the endomorphism ring of $H^1(Y(K_pK^p),\mathcal{O})$, equipped with its p-adic topology. The inverse limit

$$\widetilde{\mathbb{H}} := \varprojlim_{K_p} \widetilde{\mathbb{H}}(K_p),$$

equipped with the projective limit topology is a semi-local and noetherian ring, in particular, it has only finitely many maximal ideals. It acts faithfully on $\widetilde{H}^1(K^p,\mathcal{O})$ and we have decompositions

$$\widetilde{H}^1(K^p,\mathcal{O}) = \bigoplus_{\mathfrak{m}\in \mathrm{MaxSpec}(\widetilde{\mathbb{H}})} \widetilde{H}^1(K^p,\mathcal{O})_{\mathfrak{m}} \tag{2}$$

and

$$\widetilde{H}^1(K^p) = \bigoplus_{\mathfrak{m}} \widetilde{H}^1(K^p)_{\mathfrak{m}}$$

where in the last line

$$\widetilde{H}^1(K^p)_{\mathfrak{m}} := \widetilde{H}^1(K^p,\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} E.$$

The eigenvarieties we are working with are defined relative to the so called weight space: Let $T \subset \mathrm{SL}_2(\mathbb{Q}_p)$ be the diagonal torus and let $\widehat{T}$ be the weight space as in [6, Section 6.4]. This is the rigid analytic space over E that parametrizes the locally $\mathbb{Q}_p$-analytic characters of the group T, i.e., for any affinoid E-algebra A we have

$$\widehat{T}(A) = \mathrm{Hom}_{la}(T(\mathbb{Q}_p), A^*).$$

Emerton's eigenvarieties p-adically interpolate systems of Hecke eigenvalues coming from cohomological automorphic representations.

Let $W = \mathrm{Sym}^k(E^2)$ be the algebraic representation over E of SL_2 of highest weight k. Note that W is self-dual. Define

$$H^1(K^p, W) = \varinjlim_{K_p} H^1(Y(K_pK^p),\mathcal{W}),$$

where $\mathcal{W}$ is the local system on $Y(K_pK^p)$ attached to W. Let

$$H^1(W) = \varinjlim_{K_f} H^1(Y(K_f),\mathcal{W}),$$

where the direct limit runs over all compact open subgroups $K_f \subset \mathrm{SL}_2(\mathbb{A}_f)$.

Let π_f be an irreducible $\mathrm{SL}_2(\mathbb{A}_f)$-representation appearing as a subquotient of $H^1(W)\otimes_E \overline{\mathbb{Q}}_p$ and such that π_p embeds into the parabolic induction $\mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{Q}_p)}(\chi)$ from the Borel subgroup B of upper triangular matrices and for some smooth $\overline{\mathbb{Q}}_p$-valued character χ of T. Assume $\pi_f^{K^p}\neq 0$. Since π_f is irreducible the spherical Hecke algebra $\mathcal{H}(K^p)^{\mathrm{ur}}$ acts on π_f^p via a $\overline{\mathbb{Q}}_p$-valued character λ. Let ψ_W denote the highest weight character of W regarded as a character of T. The pair $(\chi\psi_W,\lambda)$ then defines a point of the locally ringed space $\hat{T}\times\operatorname{Spec}\mathcal{H}(K^p)^{\mathrm{ur}}$. Following Emerton we call such a point a classical point. This is justified as the representation π_f giving rise to such a point is the finite part of an automorphic representation π of SL_2. We remark at this point that as we only work with algebraic automorphic representation the finite part π_f of such a representation can always be defined over $\overline{\mathbb{Q}}$ and in fact over a number field.

In [8], Emerton constructs an eigenvariety of tame level K^p for the $\mathcal{H}(K^p)\times\mathrm{SL}_2(\mathbb{Q}_p)$ module $\widetilde{H}^1(K^p)$. We can apply his construction to any of spaces

$$\widetilde{H}^1(K^p)_{\mathfrak{m}}$$

occurring in the decomposition (2) above. The outcome of his construction is a possibly non-reduced eigenvariety $E^1(K^p,\mathfrak{m})$ equipped with a coherent sheaf $\mathcal{M}^1$ of $\mathcal{H}(K^p)^{\mathrm{ram}}$-modules. We pass to the reduced space $E(K^p,\mathfrak{m}):=E^1(K^p,\mathfrak{m})^{red}$ and let $\mathcal{M}$ be the pullback of $\mathcal{M}^1$ to $E(K^p,\mathfrak{m})$. Emerton's construction furthermore provides us with an injective map of locally ringed spaces

$$E(K^p,\mathfrak{m})\to\hat{T}\times\operatorname{Spec}\mathcal{H}(K^p)^{\mathrm{ur}}.$$

Theorem 5.1 ([8], [12]). *The eigenvariety $E(K^p,\mathfrak{m})$ is a reduced rigid analytic space which has the following properties.*

1. *Projection onto the first factor induces a finite map $E(K^p,\mathfrak{m})\to\hat{T}$.*
2. *If (ϕ,λ) is a point on $E(K^p,\mathfrak{m})$ such that ϕ is locally algebraic and of non-critical slope (in the sense of* [7, Definition 4.4.3]*), then (ϕ,λ) is a classical point.*
3. *The coherent sheaf $\mathcal{M}$ of $\mathcal{H}(K^p)^{\mathrm{ram}}$-modules over $E(K^p,\mathfrak{m})$ satisfies the following property. For any classical point $(\chi\psi_W,\lambda)\in E(K^p,\mathfrak{m})$ of non-critical slope, the fibre of $\mathcal{M}$ over the point $(\chi\psi_W,\lambda)$ is isomorphic as a $\mathcal{H}(K^p)^{\mathrm{ram}}$-module to the dual of the (χ,λ)-eigenspace of the Jacquet module of the smooth representation $H^1(K^p,W)$.*

Proof. This follows from Theorem 0.7 of [8]. The $\mathrm{SL}_2(\mathbb{A}_f)$-equivariant edge map

$$H^1(W)\otimes_E W\to\widetilde{H}^1_{W-\mathrm{lalg}}$$

from [8] Corollary 2.2.18 and Remark 2.2.19 is an isomorphism by [12]. ☐

For a classical point $z=(\chi\psi_W,\lambda)\in E(K^p,\mathfrak{m})(\overline{\mathbb{Q}}_p)$ coming from a cuspidal automorphic representation $\pi(z)$ define the classical subspace

$$\mathcal{M}^{cl}_{\overline{z}}:=\operatorname{Hom}(J_B(H^1(K^p,W))^{(\chi,\lambda)},k(z))\subset\mathcal{M}_{\overline{z}}.$$

By the results recalled in Section 2 we have

$$\begin{aligned} & J_B(H^1(K^p, W))^{(\chi,\lambda)} \otimes_{k(z),\iota} \mathbb{C} \\ \cong\ & J_B(\varinjlim_{K_p} H^1(K_pK^p, W)^\lambda)^\chi \otimes_{k(z),\iota} \mathbb{C} \\ \cong\ & J_B(\varinjlim_{K_p} H^1_{par}(K_pK^p, W)^\lambda)^\chi \otimes_{k(z),\iota} \mathbb{C} \\ \cong\ & \bigoplus_{\substack{\pi \text{ adm. rep.:} \\ \pi_l \cong \pi(z)_l \\ \forall l \notin S(K^p)\cup\{p\}}} m(\pi)(\pi_{S(K^p)})^{K^{\mathrm{ram}}} \otimes J_B(\pi_p)^\chi \otimes H^1_{rel.Lie}(\mathfrak{g}, K_\infty, W_\mathbb{C} \otimes \pi_\infty). \end{aligned}$$

In the last line we have abused notation, e.g., as we have written χ instead of $\iota \circ \chi$. We will continue to suppress it from the notation when we change coefficients from $\overline{\mathbb{Q}}_p$ to $\mathbb{C}$ using the isomorphism ι as it will always be clear from the context.

In the next section we work with certain direct summands of the sheaf $\mathcal{M}$. For that first note the following easy lemma.

Lemma 5.2. *Let $\mathcal{M}$ be a coherent $\mathcal{H}(K^p)^{ram}$-module on a rigid space X. Let $e \in \mathcal{H}(K^p)^{ram}$ be an idempotent. Then $e\mathcal{M}$ is coherent.*

Proof. As e is an idempotent we have $\mathcal{M} \cong e\mathcal{M} \oplus (1-e)\mathcal{M}$, and direct summands of coherent sheaves are coherent. □

The idempotents we use arise as follows. The subgroups $K_1(m_l)$ and $K_0(m_l)$ of $\mathrm{SL}_2(\mathbb{Z}_l)$ were defined in Section 3 for any integer $m_l \geq 0$. Recall that any character

$$\eta_l : \mathbb{Z}_l^*/(1 + l^{m_l}\mathbb{Z}_l) \to \overline{\mathbb{Q}}_p^*$$

defines a character of $K_0(m_l)$, by

$$\eta_l \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \eta_l(d).$$

Now let $K^p \subset \mathrm{SL}_2(\mathbb{A}_f^p)$ be a tame level such that $K_l = K_1(m_l)$ for all $l \in S(K^p)$. Choose characters η_l as above and let

$$\eta := \prod_{l \in S(K^p)} \eta_l : \prod_l K_0(m_l) \to \overline{\mathbb{Q}}_p^*$$

be their product. Then if $E \subset \overline{\mathbb{Q}}_p$ contains the values of η, there exists an idempotent e_η in $\mathcal{H}(K^p)^{\mathrm{ram}}$ associated with the representation η of $K_0^{\mathrm{ram}} := \prod_{l \in S(K^p)} K_0(m_l)$. Concretely we choose the Haar measure on $\mathrm{SL}_2(\mathbb{A}_{S(K^p)})$ such that $\mu_l(K_0(m_l)) = 1$ for all $l \in S(K^p)$. Then let $e_\eta : \mathrm{SL}_2(\mathbb{A}_{S(K^p)}) \to E$ be defined as

$$e_\eta(g) = \begin{cases} \eta^{-1}(g), & \text{if } g \in K_0^{\mathrm{ram}} \\ 0, & \text{otherwise.} \end{cases}$$

One easily checks that this defines an idempotent and that for two distinct characters $\eta \neq \eta'$ the resulting idempotents are orthogonal. For any smooth representation π of $\mathrm{SL}_2(\mathbb{A}_{S(K^p)})$ with coefficients in $\overline{\mathbb{Q}}_p$, e_η is the projection onto the space $\pi_\eta^{K_0^{\mathrm{ram}}}$.

In the next section we need some extra properties that the eigenvarieties $E(K^p, \mathfrak{m})$ satisfy, at least when the maximal ideal $\mathfrak{m}$ is suitably chosen. In the rest of this section we will explain and verify these properties.

Recall that a rigid space $X/\operatorname{Sp}(E)$ is called nested if it has an admissible cover by open affinoid subspaces $\{X_i\}_{i\geq 0}$ such that $X_i \subset X_{i+1}$ for all $i \geq 0$ and the natural E-linear map $\mathcal{O}(X_{i+1}) \to \mathcal{O}(X_i)$ is compact.

Lemma 5.3. *The eigenvariety $E(K^p, \mathfrak{m})$ is nested. In particular,*

$$\mathcal{O}(E(K^p, \mathfrak{m}))^0 := \{f \in \mathcal{O}(E(K^p, \mathfrak{m})) : |f(x)| \leq 1 \ \forall x \in E(K^p, \mathfrak{m})\}$$

is compact.

Proof. The space $\hat{T} \cong \operatorname{Hom}_{la}(T(\mathbb{Z}_p), \mathbb{G}_m) \times \mathbb{G}_m$ is a product of nested spaces, therefore nested. The map $E(K^p) \to \hat{T}$ is finite. By [2, Lemma 7.2.11] this implies that $E(K^p)$ is also nested. As $E(K^p)$ is reduced this implies the compactness assertion, again by [2, Lemma 7.2.11]. □

Definition 5.4. *Define the ideal $\mathfrak{m}_0$ to be the kernel of the map*

$$\mathbb{H} \to \mathbb{F}_q$$

that sends t_l to $l^2 + l$.

Remark 5.5. *This ideal comes from the fact that $t_l \in \mathbb{H}$ acts on $H^0(K^p, \mathcal{O}) \cong \mathcal{O}$ by $x \mapsto (l^2 + l)x$, in particular*

$$H^0(K^p, \mathcal{O})_{\mathfrak{m}} = 0,$$

for all $\mathfrak{m} \in \operatorname{MaxSpec}(\mathbb{H}), \mathfrak{m} \neq \mathfrak{m}_0$.

For an open subgroup $H \subset \mathrm{SL}_2(\mathbb{Q}_p)$ let $\mathcal{C}^{la}(H, E)$ denote the locally analytic E-valued functions on H.

Lemma 5.6. *Let $\mathfrak{m} \in \operatorname{MaxSpec}(\mathbb{H}), \mathfrak{m} \neq \mathfrak{m}_0$ be a maximal ideal with $\widetilde{H}^1(K^p, \mathcal{O})_{\mathfrak{m}} \neq 0$. There is a compact open subgroup $H \subset \mathrm{SL}_2(\mathbb{Q}_p)$ such that*

$$(\widetilde{H}^1(K^p, \mathcal{O})_{\mathfrak{m}})_{la} \xrightarrow{\sim} \mathcal{C}^{la}(H, E)^r,$$

for some $r \geq 1$ as representations of H.

Proof. This is proved as in [9]. We have $\widehat{H}^1(K^p, \mathcal{O}) \cong \widetilde{H}^1(K^p, \mathcal{O})$, and

$$\widehat{H}^1(K^p, \mathcal{O})_{\mathfrak{m}} / \varpi^s \widehat{H}^1(K^p, \mathcal{O})_{\mathfrak{m}} \cong H^1(K^p, \mathcal{O}/\varpi^s \mathcal{O})_{\mathfrak{m}}.$$

By the proof of Corollary 5.3.19 in [9] it suffices to show that we can find an open subgroup H such that $H^1(K^p, \mathcal{O}/\varpi^s \mathcal{O})_{\mathfrak{m}}$ is injective for all $s \geq 1$. This last claim can be proved in the same way as Proposition 5.3.15 in [9]. Our assumption $\mathfrak{m} \neq \mathfrak{m}_0$ implies that all the relevant H^0-terms vanish. □

Lemma 5.7. *Let $\mathfrak{m} \neq \mathfrak{m}_0$ be a maximal ideal in $\mathbb{H}$ as above. The eigenvariety $E(K^p, \mathfrak{m})$ is equidimensional of dimension one. It contains a Zariski-dense set of classical points, which accumulates at any of its points.*

Proof. Define $\widehat{T}_0 := \mathrm{Hom}_{la}(T(\mathbb{Z}_p), \mathbb{G}_m)$, [2] then $\widehat{T} \cong \widehat{T}_0 \times_E \mathbb{G}_m$. Write $\widehat{T}_0$ as an increasing union of affinoid opens $\mathrm{Sp}(A_n)$, $n \geq 0$.

Lemma 5.6 allows us to apply the results from the proof of Proposition 4.2.36 in [7]. In particular, we get that locally over $\mathrm{Sp}(A_n) \subset \widehat{T}_0$, there is a finite map

$$E(K^p, \mathfrak{m})_n \to Z_n$$

from the eigenvariety $E(K^p, \mathfrak{m})_n := E(K^p, \mathfrak{m}) \times_{\widehat{T}_0} \mathrm{Sp}(A_n)$ to the spectral variety $Z_n \hookrightarrow \mathbb{G}_m \times_E \mathrm{Sp}(A_n)$ attached to the operator $u_p := \begin{pmatrix} p & \\ & p^{-1} \end{pmatrix} \in T(\mathbb{Q}_p)$ which acts compactly on the respective Banach space. We refer to the proof of Corollary 4.1 in [18], where a similar argument is explained nicely in the context of GL_2. In fact reinterpreting all the GL_2-specific notation occurring there in terms of SL_2 and replacing the representation V by $\widetilde{H}^1(K^p)_{\mathfrak{m}}$, everything goes through. In particular, for every point on $E(K^p, \mathfrak{m})$ we may find an arbitrary small open affinoid neighbourhood U and an open subspace $V \subset \widehat{T}_0$ such that the map

$$U \to V$$

obtained as the restriction of $E(K^p, \mathfrak{m}) \to \widehat{T}_0$ to U, is finite and surjective when restricted to each irreducible component. We may therefore deduce the Zariski-density and accumulation property as in [5, Section 6] from the fact that the classical weights $\{k \geq 2\} \subset \widehat{T}_0$ are Zariski-dense and accumulation in weight space $\widehat{T}_0$. □

From Theorem 5.1 above we get a morphism $\psi : \mathcal{H}(K^p)^{\mathrm{ur}} \to \mathcal{O}(E(K^p, \mathfrak{m}))$. Abbreviate $S := S(K^p)$. Let $t_l \in \mathcal{H}(K^p)^{\mathrm{ur}}$ be the characteristic function on the double coset

$$\mathrm{SL}_2(\widehat{\mathbb{Z}}^S) \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix} \mathrm{SL}_2(\widehat{\mathbb{Z}}^S),$$

where $\begin{pmatrix} l & \\ & l^{-1} \end{pmatrix}$ is understood to be the matrix in $\mathrm{SL}_2(\widehat{\mathbb{Z}}^S) = \prod_{q \notin S} \mathrm{SL}_2(\mathbb{Z}_q)$ which is equal to 1 for all $q \neq l$ and equal to $\begin{pmatrix} l & \\ & l^{-1} \end{pmatrix}$ at l.

Lemma 5.8. *Let $E(K^p, \mathfrak{m})$ be the eigenvariety of tame level K^p. Let $G_{\mathbb{Q},S}$ be the Galois group of a maximal extension of $\mathbb{Q}$ unramified outside S. There exists a three dimensional pseudorepresentation*

$$T : G_{\mathbb{Q},S} \to \mathcal{O}(E(K^p, \mathfrak{m}))$$

such that $T(\mathrm{Frob}_l) = \psi\left(\frac{1}{l}(t_l + 1)\right)$, for all $l \notin S$.

Proof. We follow the strategy of Proposition 7.1.1 in [5]. The fact that

$$\psi\left(\frac{1}{l}(t_l + 1)\right) \in \mathcal{O}(E(K^p, \mathfrak{m}))^0$$

comes from the definition of the action of $\mathcal{H}(K^p)^{ur}$ on $\widetilde{H}^1(K^p)_{\mathfrak{m}}$ via correspondences. Then Lemma 5.7 implies that we may apply [5, Prop.7.1.1] to the Zariski-dense subset of classical points and the three-dimensional pseudorepresentations that are described in the proof of Lemma 2.2 of [17]. □

[2]This is the weight space considered in the work of Coleman and Buzzard.

Lemma 5.9. *Let T be a pseudorepresentation on $E(K^p, \mathfrak{m})$. Then for $l \in S$, $T|_{I_l}$ is constant on connected components of $E(K^p, \mathfrak{m})$.*

Proof. [2] Lemma 7.8.17. □

Proposition 5.10. *Let $x \in E(K^p, \mathfrak{m})(\overline{\mathbb{Q}}_p)$ be a classical point and choose an automorphic representation π_x which gives rise to x. Assume for some $l \neq p$, $\pi_{x,l}$ is supercuspidal and that the L-packet of $\pi_{x,l}$ contains exactly two elements. Let $y \in E(K^p, \mathfrak{m})(\overline{\mathbb{Q}}_p)$ be another classical point which is on the same connected component as x. Then any automorphic representation π_y giving rise to y has the property that $\pi_{y,l}$ is supercuspidal, and the L-packet containing $\pi_{y,l}$ is again of size two.*

Proof. Choose lifts $\widetilde{\pi}_x$ and $\widetilde{\pi}_y$ to GL_2 that are unramified at all places not in S. Then T_x is the trace of the representation $\tau_x := \mathrm{Sym}^2(\rho(\widetilde{\pi}_x)) \otimes \det(\rho(\widetilde{\pi}_x))^{-1}$, where $\rho(\widetilde{\pi}_x)$ is the 2-dimensional Galois representation associated to $\widetilde{\pi}_x$ by Deligne and similarly for y. As $T_x|_{I_l} = T_y|_{I_l}$, the lifts $\mathrm{rec}^{-1}(\tau_{x,l})$ of $\widetilde{\pi}_{x,l}$ and $\mathrm{rec}^{-1}(\tau_{y,l})$ of $\widetilde{\pi}_{y,l}$ to $\mathrm{GL}_3(\mathbb{Q}_l)$ are in the same Bernstein component and we can apply Proposition 4.2 to get the result. □

6. Existence of non-classical p-adic automorphic forms

Let $p \geq 7$ be a prime. Let $\Pi(\theta)$ be an endoscopic L-packet attached to an algebraic character

$$\theta : \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m(\mathbb{A}) \to \mathbb{C}^*$$

of the adelic points of the elliptic endoscopic group $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$, where $F/\mathbb{Q}$ is an imaginary quadratic field in which p splits. Let S be the set of finite primes l, where $\Pi(\theta)_l$ is ramified. We say that $\Pi(\theta)$ satisfies *Hypothesis* ($\star$) if

1. $\Pi(\theta)_p$ is a singleton consisting of a $\mathrm{SL}_2(\mathbb{Z}_p)$-unramified representation,
2. for all $l \in S$, $\Pi(\theta)_l$ is a supercuspidal L-packet of size two,
3. $2 \notin S$ and
4. $\Pi(\theta)_\infty = \{D^\pm_{k+1}\}$ is a discrete series L-packet of weight $k + 1 \geq 1$.

We refer the reader to [17, Lemma 4.1] for the construction of examples of such L-packets.

Lemma 6.1. *Let $\tau_f \in \Pi(\theta)_f$ be any element. Then exactly one of the representations $\tau_f \otimes D^+_{k+1}$ and $\tau_f \otimes D^-_{k+1}$ is automorphic.*

Proof. This follows from the multiplicity formulae in [13]; more precisely we are in the situation of Proposition 6.7 in loc.cit. □

Let $c_l := c(\Pi(\theta)_l)$ be the conductor of the local L-packet $\Pi(\theta)_l$ as in Section 3. From now on we fix the tame level to be

$$K^p := \prod_{l \in S} K_1(c_l) \times \prod_{l \notin S \cup \{p\}} \mathrm{SL}_2(\mathbb{Z}_l) \subset \mathrm{SL}_2(\mathbb{A}^p_f).$$

We also fix a coefficient field $E/\overline{\mathbb{Q}}_p$ big enough to contain the values of all characters η occurring below. For the main theorem it suffices that E contains the $(l-1)$th roots of unity for all $l \in S$.

Furthermore we fix $\tau_f \in \Pi(\theta)_f$ such that $(\tau_f^p)^{K^p} \neq 0$. Let

$$\tau' \in \{\tau_f \otimes D_{k+1}^+, \tau_f \otimes D_{k+1}^-\} \tag{3}$$

be the unique representation with $m(\tau') = 1$. Then τ' gives rise to a maximal ideal $\mathfrak{m} \subset \mathbb{H}$ as follows: Let $\widetilde{\tau}'$ be a lift of τ' to $\mathrm{GL}_2(\mathbb{A})$ which is unramified at all $l \notin S$. Attached to $\widetilde{\tau}'$ there is a two dimensional Galois representation $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ constructed by Deligne. Let $\overline{\rho}$ be the reduction of $\rho \mod p$. Define $\mathfrak{m}$ to be the kernel of the map

$$\mathbb{H} \to \overline{\mathbb{F}}_p,\ t_l + 1 \mapsto l\frac{\mathrm{tr}^2(\overline{\rho}(\mathrm{Frob}_l))}{\det(\overline{\rho}(\mathrm{Frob}_l))}.$$

Lemma 6.2. *The ideal $\mathfrak{m}$ is not equal to $\mathfrak{m}_0$.*

Proof. The representation ρ is induced from a character of G_F. Therefore the trace of $\overline{\rho}$ vanishes for all primes l that are inert in F, which is of density $1/2$. On the other hand the density of primes l such that $l^2+l+1 = 0 \mod p$ is $2/(p-1) < 1/2$, as $p \geq 7$. The claim follows. □

Let $z = (\chi\psi_W, \lambda) \in E(K^p, \mathfrak{m})(\overline{\mathbb{Q}}_p)$ be any classical point coming from an automorphic representation π. We denote by $\Pi(z)$ the global L-packet containing π. Note that this is well-defined as λ determines a unique representation π_l of $\mathrm{SL}_2(\mathbb{Q}_l)$ at all places $l \notin S$ and for SL_2 this is enough to determine the global L-packet.

Definition 6.3. *Let z be a classical point on $E(K^p, \mathfrak{m})$. We say z is* stable *if the L-packet $\Pi(z)$ defined by z is a stable L-packet. Otherwise we call the point an* endoscopic *point.*

The representation τ' from (3) above gives rise to two distinct points $x, y \in E(K^p, \mathfrak{m})(\overline{\mathbb{Q}}_p)$, one of which, say x, is of critical slope whereas the point y is of non-critical slope in the sense of [7, Def.4.4.3]. One verifies this easily using [7, Lemma 4.4.1], a calculation like in Lemma 3.3 of [17] and the fact that $\tau_p \cong \mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{Q}_p)}(\theta_w\theta_{\overline{w}}^{-1})$, where w and $\overline{w}$ denote the two places in F above p.

We say that a global L-packet Π has property $\mathcal{P}(S)$ if

- the local L-packet Π_l is unramified for all $l \notin S$,
- the local L-packet Π_l is supercuspidal of size two for all $l \in S$.

By Lemma 5.10 any classical point $z \in E(K^p, \mathfrak{m})$ on the same connected component as x has property $\mathcal{P}(S)$.

For a L-packet Π we let $\omega_{\Pi_S} : \prod_{l\in S} Z(\mathrm{SL}_2(\mathbb{Q}_l)) \to \{\pm 1\}$ be the product of the central characters of the representations $\pi_l \in \Pi_l$ for $l \in S$. The following proposition is the key calculation for our main theorem. Define $c := \prod_{l\in S} c_l$, , where $c_l := c(\Pi(\theta)_l)$ as before and let $K_0(c) := \prod_{l\in S} K_0(c_l)$.

Proposition 6.4. *(1) Let $z \in E(K^p, \mathfrak{m})(\overline{\mathbb{Q}}_p)$ be a stable point on the same connected component as the point x. Let $\Pi(z)$ be its associated L-packet. Let $\eta = \prod \eta_l : \prod_{l\in S} \mathbb{Z}_l^* \to \overline{\mathbb{Q}}_p^*$ be a character such that $c(\eta_l) \leq c(\Pi(z)_l)$ for all*

$l \in S$. Then if $\eta(-1) = \omega_{\Pi(z)_S}(-1)$ we have

$$\dim_{k(z)} e_\eta \mathcal{M}^{cl}_{\overline{z}} \geq 2(\prod_{l\in S} 2(c_l - c(\Pi(z)_l) + 1)).$$

Otherwise $e_\eta \mathcal{M}^{cl}_{\overline{z}} = 0$.

(2) Let $\eta = \prod \eta_l : \prod \mathbb{Z}_l^* \to \overline{\mathbb{Q}}_p^*$ be a character such that $c(\eta_l) \leq c_l$ for all $l \in S$. Then

$$\dim_{k(x)} e_\eta \mathcal{M}^{cl}_{\overline{x}} = 2^{\# S}$$

if $\eta(-1) = \omega_{\Pi(\theta)_S}(-1)$. Otherwise $e_\eta \mathcal{M}^{cl}_{\overline{x}} = 0$.

Proof. Given any classical point $z \in E(K^p, \mathfrak{m})(\overline{\mathbb{Q}}_p)$ there is by definition an automorphic representation $\pi = \otimes \pi_l \in \Pi(z)$ with $(\pi_f^p)^{K^p} \neq 0$, and so for all $l \in S$, there exists η_l' such that $(\pi_l)^{K_0(c_l)}_{\eta_l'} \neq 0$. Therefore $c(\Pi(z)_l) \leq c_l$. First note that for $W = \mathrm{Sym}^k(E^2)$, we have

$$\begin{aligned} & J_B(H^1_{\mathrm{par}}(K^p, W)) \otimes_{E,\iota} \mathbb{C} \\ = \quad & J_B\left(\varinjlim_{K_p} \bigoplus_{\substack{\pi \ adm.\ rep. \\ of\, \mathrm{SL}_2(\mathbb{A})}} m(\pi) \pi_f^{K^p K_p} \otimes H^1_{rel.Lie}(\mathfrak{g}, K_\infty, W_\mathbb{C} \otimes \pi_\infty) \right) \\ = \quad & \bigoplus_{\pi \ adm.\ rep} m(\pi) (\pi_f^p)^{K^p} \otimes J_B(\pi_p) \otimes H^1_{rel.Lie}(\mathfrak{g}, K_\infty, W_\mathbb{C} \otimes \pi_\infty). \end{aligned}$$

By Lemma 2.1 above, the term

$$H^1_{rel.Lie}(\mathfrak{g}, K_\infty, W_\mathbb{C} \otimes \pi_\infty)$$

is non-zero if and only if $\pi_\infty = D^\pm_{k+1}$ and in this case it is isomorphic to $\mathbb{C}$. Therefore for any classical point $z = (\chi \psi_W, \lambda) \in E(K^p, \mathfrak{m})(\overline{\mathbb{Q}}_p)$ we have

$$J_B(H^1(K^p, W))^{(\chi,\lambda)} \otimes_{k(z),\iota} \mathbb{C} = \bigoplus_{\substack{\pi \ adm.\ rep. \\ \pi_\infty = D^\pm_{k+1} \\ \pi^{S\cup\{p\}} \cong \pi_\lambda}} m(\pi) (\pi_S)^{K^{\mathrm{ram}}} \otimes J_B(\pi_p)^{\iota(\chi)},$$

where $k(z)$ is the residue field at z and π_λ is the representation of $\mathrm{SL}_2(\mathbb{A}_f^{S\cup\{p\}})$ determined by $\iota \circ \lambda$. Here and below we write $\iota(\chi)$ for the composition $\iota \circ \chi$.

We see that the direct sum runs over a subset of the global L-packet $\Pi(z)$ determined by z, namely over

$$X(z) := \{\pi \in \Pi(z) | \pi_l = \pi_l^0 \text{ for all } l \notin S \cup \{p\}\}.$$

Here π_l^0 denotes the unique member of $\Pi(z)_l$, which has a fixed vector under $\mathrm{SL}_2(\mathbb{Z}_l)$. The following observation is crucial: If z is a stable point then any $\pi \in X(z)$ is an automorphic representation, whereas for the point x the number of automorphic representations in $X(x)$ is $\# X(x)/2$.

Now let η be as in the statement of the proposition and let $z = (\chi\psi_W, \lambda) \in E(K^p, \mathfrak{m})$ be a stable point on the connected component of x. Then

$$\begin{aligned}
\dim_{k(z)} e_\eta \mathcal{M}^{cl}_{\overline{z}} &= \dim_{k(z)} e_\eta \operatorname{Hom}(J_B(H^1(K^p, W))^{(\chi,\lambda)}, k(z)) \\
&= \sum_{\pi \in X(z)} m(\pi) \dim_{\mathbb{C}} \left(\iota(e_\eta)\left((\pi_{\mathrm{ram}})^{K^{\mathrm{ram}}} \otimes J_B(\pi_p)^{\iota(\chi)}\right)\right) \\
&= \sum_{\pi \in X(z)} \dim_{\mathbb{C}} \left(\left(\iota(e_\eta)(\pi_{ram})^{K^{ram}}\right) \otimes J_B(\pi_p)^{\iota(\chi)}\right) \\
&= \sum_{\pi \in X(z)} \left(\prod_{l \in S} \dim_{\mathbb{C}} (\pi_l)^{K_0(c_l)}_{\iota(\eta_l)}\right) \dim_{\mathbb{C}} J_B(\pi_p)^{\iota(\chi)},
\end{aligned}$$

where in the second to last line we have used that any element $\pi \in X(z)$ is automorphic. Define

$$X(z)^+ = \{\pi \in X(z) : \pi_\infty = D^+_{k+1}\}$$

and $X(z)^- = \{\pi \in X(z) : \pi_\infty = D^-_{k+1}\}$. Then

$$\begin{aligned}
&\dim_{k(z)} e_\eta \mathcal{M}^{cl}_{\overline{z}} \\
= &\sum_{\pi \in X(z)^+} \left(\prod_{l \in S} \dim_{\mathbb{C}} (\pi_l)^{K_0(c_l)}_{\iota(\eta_l)}\right) \dim_{\mathbb{C}} J_B(\pi_p)^{\iota(\chi)} \\
&+ \sum_{\pi \in X(z)^-} \left(\prod_{l \in S} \dim_{\mathbb{C}} (\pi_l)^{K_0(c_l)}_{\iota(\eta_l)}\right) \dim_{\mathbb{C}} J_B(\pi_p)^{\iota(\chi)} \\
\geq &\ 2\left(\prod_{l \in S} 2(c_l - c(\Pi(z)_l) + 1)\right) \geq 2^{\# S+1}.
\end{aligned}$$

where in the last line we have used Proposition 3.4. Furthermore the second to last inequality comes from the fact that $\Pi(z)_p$ might be of size two.

We now calculate the corresponding space at the point $x = (\chi(x)\psi_{W(x)}, \lambda(x)) \in E(K^p, \mathfrak{m})(\overline{\mathbb{Q}}_p)$. Let $X(x)_{aut} \subset X(x)$ be the subset of automorphic representations. The key point in the following is that $X(x)_{aut} \neq X(x)$. We have

$$\begin{aligned}
\dim_{k(x)} e_\eta \mathcal{M}^{cl}_{\overline{x}} &= \dim_{\mathbb{C}} \left(e_\eta J_B(H^1(K^p, W(x)))^{(\chi(x),\lambda(x))} \otimes_{k(x)} \mathbb{C}\right) \\
&= \sum_{\pi \in X(x)} m(\pi) \dim_{\mathbb{C}} \left(\left(\iota(e_\eta)(\pi_{ram})^{K^{ram}}\right) \otimes J_B(\pi_p)^{\iota(\chi(x))}\right).
\end{aligned}$$

By our assumptions the L-packet $\Pi(\theta)_p = \{\pi(\theta)_p\}$ is a singleton and by Lemma 2.2

$$\dim_{\mathbb{C}} J_B(\pi(\theta)_p)^{\iota(\chi(x))} = 1.$$

Therefore

$$\begin{aligned}\dim e_\eta \mathcal{M}^{cl}_{\overline{x}} &= \sum_{\pi\in X(x)_{aut}} \prod_{l\in S} \dim(\pi_l)^{K_0(c_l)}_{\iota(\eta_l)} \\ &= \prod_{l\in S} \sum_{\pi_l\in\Pi_l} \dim(\pi_l)^{K_0(c_l)}_{\iota(\eta_l)} \\ &= \prod_{l\in S} 2 = 2^{\#S}.\end{aligned}$$

□

Theorem 6.5. *Let $\Pi(\theta)$ be an endoscopic L-packet of $\mathrm{SL}_2(\mathbb{A})$ satisfying Hypothesis $(\star)$. Let $\tau_f \in \Pi(\theta)_f$ be such that $\tau_f^{K^p} \neq 0$ and let x be the point on $E(K^p,\mathfrak{m})$ of critical slope defined by the automorphic representation $\tau' \in \Pi(\theta)$ with $\tau'_f = \tau_f$. Then there exist non-zero non-classical forms in $\mathcal{M}_{\overline{x}}$, i.e.,*

$$\mathcal{M}_{\overline{x}}/\mathcal{M}^{cl}_{\overline{x}} \neq 0.$$

Proof. Fix an affinoid open neighbourhood U of x, such that U contains a Zariski-dense set Z of classical points. Such a neighbourhood exists by Lemma 5.7, whose proof also provides us with the compact operator u_p. After possibly shrinking U we may assume that the slope of u_p is constant on U. By an argument as in the proof of [17, Theorem 4.3] we may assume that $\Pi(z)$ is stable for all $z \in Z, z \neq x$.

Let $\mathcal{X}$ be the set of characters

$$\prod_{l\in S} \mathrm{Hom}(\mathbb{Z}_l^*/(1+l\mathbb{Z}_l), \overline{\mathbb{Q}}_p^*) \cong \prod_{l\in S} \mathbb{F}_l^*.$$

As before we view $\eta \in \mathcal{X}$ as a character of the group $K_0(c)$. Define

$$\mathcal{M}' := \bigoplus_{\eta\in\mathcal{X}} e_\eta \mathcal{M}.$$

By Lemma 5.2 this is a coherent sheaf on $E(K^p,\mathfrak{m})$ and a direct summand of $\mathcal{M}$. For any classical point $z \in E(K^p,\mathfrak{m})(\overline{\mathbb{Q}}_p)$, let

$$(\mathcal{M}'_{\overline{z}})^{cl} := \mathcal{M}^{cl}_{\overline{z}} \cap \mathcal{M}'_{\overline{z}} = \bigoplus_{\eta\in\mathcal{X}} e_\eta \mathcal{M}^{cl}_{\overline{z}}.$$

To prove the theorem it suffices to show that there exist non-classical forms in $\mathcal{M}'_{\overline{x}}$.

Given any character

$$\mu : Z_S := \prod_{l\in S} Z(\mathrm{SL}_2(\mathbb{Q}_l)) \to \overline{\mathbb{Q}}_p^*$$

there are exactly $n := \prod_{l\in S}(l-1)/2$ elements $\eta \in \mathcal{X}$ such that $\eta|_{Z_S} = \mu$.

Therefore by Proposition 6.4 we have

$$\dim_{k(x)} (\mathcal{M}'_{\overline{x}})^{cl} = n2^{\#S}.$$

On the other hand let $z \in Z$ be a stable point in U, then Proposition 6.4 implies that

$$\dim_{k(z)} (\mathcal{M}'_{\overline{z}})^{cl} \geq n2^{\#S+1}.$$

As Z is Zariski-dense in U and $\mathcal{M}'$ is coherent, the semi-continuity of the fibre rank implies that

$$\dim_{k(x)} \mathcal{M}'_{\overline{x}} \geq n2^{\#S+1}.$$

Therefore

$$\mathcal{M}'_{\overline{x}}/(\mathcal{M}'_{\overline{x}})^{cl} \neq 0.$$

□

Remark 6.6. *(1) There is an analogous situation on the Coleman-Mazur eigencurve: For a modular eigenform $f \in S_k(\Gamma_1(N))$ of weight $k \geq 2$ with complex multiplication by F, the generalized eigenspace*

$$\mathcal{M}^{\dagger}_k(\Gamma_1(N) \cap \Gamma_0(p))_{(x_{crit})}$$

of overconvergent modular forms contains a non-classical form. For details see Proposition 2.11 and 2.13 in [1]*. Note however that it is necessary to pass to the generalized eigenspace to see non-classical forms, the corresponding eigenspace consists only of classical forms. For the group* SL_2 *our theorem shows that non-classical forms occur already in the eigenspace.*

(2) One cannot hope for an analogue of the above theorem at the other refinement: The point y is of non-critical slope and therefore $\mathcal{M}^{cl}_{\overline{y}} = \mathcal{M}_{\overline{y}}$.

(3) In the proof of the theorem above we used that there is a Zariski-dense set of stable points around the endoscopic point x. We want to briefly explain what happens for a point close to x which comes from an endoscopic packet Π *with p inert in the corresponding imaginary quadratic field. We also assume that we keep (2) to (4) in Hypothesis ($\star$) and that* Π_p *is unramified. Now as p is inert, the L-packet* Π_p *is of size two and consists of the constituents of* $\mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{Q}_p)}(\chi)$ *for a quadratic character χ. Then for any choice of π_f^p we have four elements in* Π *that agree with π_f^p at all places different from p and ∞, and two of them are automorphic. Both automorphic representations give rise to the same point* x_{inert} *on the eigenvariety and it is non-critical. Note also that $\chi = \chi^{-1}$ and so from the proof of Lemma 2.2 we can directly see that the dimension of the classical subspace* $(\mathcal{M}'_{x_{\mathrm{inert}}})^{cl}$ *is at least $n2^{\#S+1}$.*

Let $f \in \mathcal{M}_{\overline{x}}$ be a non-classical form as provided by the theorem. It would be interesting to understand the relationship between f and the packet $\Pi(\theta)$ in a more concrete way.

Furthermore, using the canonical lifting $\varphi : J_B(\widetilde{H}^1(K^p, W)_{la}) \to \widetilde{H}^1(K^p, W)_{la}$ constructed in Section (3.4.8) of [7] we get a vector $\varphi(f)$ in the $\mathrm{SL}_2(\mathbb{Q}_p)$-representation $\widetilde{H}^1(K^p, W)_{la}$. The representation $\langle \mathrm{SL}_2(\mathbb{Q}_p)\varphi(f)\rangle$ generated by this vector is of interest in the context of the p-adic local Langlands programme for $\mathrm{SL}_2(\mathbb{Q}_p)$. It would be interesting to describe this representation explicitly.

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E-mail address: `jludwig@math.uni-bonn.de`